KMITL Sci. J. Vol. 5 No. 1 Feb. 2005

# SPHERICAL CONIC AND THE FOURTH PARAMETER

Yoichi Maeda
Department of Mathematics, Tokai University, 1117 Kitakaname, Hiratsuka, Kanagawa, 259-1292, Japan
E-mail : maeda@keyaki.cc.u-tokai.ac.jp

## ABSTRACT

As the same as planar ellipse in Euclidean plane, spherical ellipse on the unit sphere is determined by two parameters: two foci and a string with certain length. Spherical ellipse is also the intersection of an elliptic cone centered at the origin and the unit sphere. Spherical ellipse has three parameters; focus, major axis, and minor axis. In this paper, we will introduce the fourth parameter, asymptotic angle. There are seven relations between these four parameters. Duality is a key word to understand these relations.

**KEYWORDS:** spherical conic, asymptotic angle, duality

# 1. INTRODUCTION

In this paper, let us investigate the spherical ellipse (spherical conic) and the relations between four parameters of the spherical ellipse. First, let us review the planar ellipse on the flat plane as Fig. 1. This ellipse is geometrically defined by two parameters, that is, two foci ($C(c,0)$ and $C'(-c,0)$) and the string with length $L$ such as the locus $P$ satisfying $CP + C'P = L$. Point $A(a,0)$ is called the major axis, and point $B(0,b)$ is called the minor axis. It is well known that

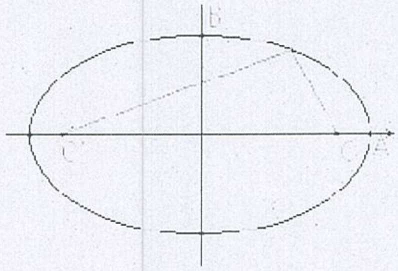


Figure 1: Ordinary Planar Ellipse

the string length $L$ is equal to the double of $a$ (set a point $P$ on the ellipse at $A$), and the Pythagorean relation $a^2 = b^2 + c^2$ (set a point $P$ on the ellipse at $B$). This ellipse is algebraically defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The purpose of this paper is to find out the similar relations on the spherical ellipse. We will introduce the fourth new parameter $d$, and there are seven interesting and amazing relations between them.

# 2. SPHERICAL ELLIPSE

Let us consider the spherical ellipse on the unit sphere. As the same as the planer ellipse, a spherical ellipse is geometrically defined by two parameters, that is, two foci ($C$ and $C'$) and the string with length $L$ such as the locus $P$ satisfying $arc\ length(CP) + arc\ length(C'P) = L$. In this case, this string is composed of two arcs on the unit sphere. Fig. 2 shows a spherical ellipse on the unit sphere centered at the north pole $N(0,0,1)$. Set $A(\sin a, 0, \cos a)$, $B(0, \sin b, \cos b)$, and $C(\sin c, 0, \cos c)$ as one of the end points of major, minor axes, and foci, respectively($0 < b < a < \pi/2$, $0 < c < a$). Note that $a = arc\ length(NA)$ is equal to the central angle $\angle NOA$. As the same as the planar ellipse, the string length $L$ is the double of $a$. In addition, the spherical triangle $\triangle BNC$ is a right-angled triangle, hence

$$\cos a = \cos b \cos c \qquad (1)$$

from the spherical cosine law for sides [2](pp.54). However, it is difficult to imagine this spherical ellipse from the geometrical definition by the foci and the

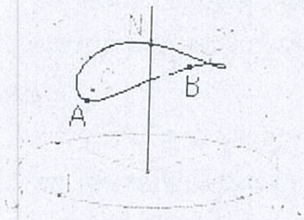


Figure 2: Spherical Ellipse on the Unit Sphere

string. Algebraically, this spherical ellipse defined as the intersection of the unit sphere and an elliptic cone as in the following theorem [1] (pp.312).

**Theorem 1.** *Spherical ellipse is given as the intersection of the unit sphere $(x^2 + y^2 + z^2 = 1)$ and a cone centered at the origin determined by the following equation:*

$$\frac{x^2}{\tan^2 a} + \frac{y^2}{\tan^2 b} = z^2 \quad (z > 0). \tag{2}$$

*Proof.* Let $P(x, y, z)$ be a point on the spherical ellipse and $C'(-\sin c, 0, \cos c)$ be another focus. Then,

$$\cos\angle COP = x\sin c + z\cos c, \qquad \sin\angle COP = \sqrt{1 - (x\sin c + z\cos c)^2},$$
$$\cos\angle C'OP = -x\sin c + z\cos c, \qquad \sin\angle C'OP = \sqrt{1 - (-x\sin c + z\cos c)^2}.$$

$\cos(\angle COP + \angle C'OP) = \cos 2a$ implies that

$$z^2\cos^2 c - x^2\sin^2 c - \cos 2a = \sqrt{(1 - (x\sin c + z\cos c)^2)(1 - (-x\sin c + z\cos c)^2)}.$$

Squaring the both side of the above equation,

$$-2\cos 2a(z^2\cos^2 c - x^2\sin^2 c) + \cos^2 2a = 1 - 2(x^2\sin^2 c + z^2\cos^2 c).$$

Consequently, we get

$$\frac{x^2}{\sin^2 a/\sin^2 c} + \frac{z^2}{\cos^2 a/\cos^2 c} = 1.$$

Now, using $\cos a = \cos b\cos c$, and $x^2 + y^2 + z^2 = 1$,

$$\frac{x^2}{\sin^2 a/\sin^2 c} + \frac{z^2}{\cos^2 b} = x^2 + y^2 + z^2.$$

As a result, we get

$$\frac{x^2}{\tan^2 a} + \frac{y^2}{\tan^2 b} = z^2.$$

**Corollary 1.** *Spherical ellipse is given as the intersection of the unit sphere ($x^2 + y^2 + z^2 = 1$) and one of three cylinders determined by the following equations:*

$$\frac{x^2}{\sin^2 a} + \frac{y^2}{\sin^2 b} = 1. \tag{3}$$

$$\frac{x^2}{\sin^2 a/\sin^2 c} + \frac{z^2}{\cos^2 b} = 1 \quad (z > 0). \tag{4}$$

$$-\frac{y^2}{\cos^2 a\tan^2 b/\sin^2 c} + \frac{z^2}{\cos^2 a} = 1 \quad (z > 0). \tag{5}$$

*Proof.* Using Equation (2) $x^2/\tan^2 a + y^2/\tan^2 b = z^2$, $x^2 + y^2 + z^2 = 1$, and Equation (1) $\cos a = \cos b\cos c$, these equations are easily derived from eliminating the term $z^2$, $y^2$, and $x^2$, respectively. □

Equation (3) means that the orthogonal projection of the spherical ellipse to xy-plane is an ellipse in the unit circle (See Fig. 3 (left)). In the same way, Equation (4) means that the projection to xz-plane is an ellipse intersecting with the unit circle (See Fig. 3 (center)). Hence the projected image is a part of the ellipse. And Equation (5) also means that the projected image to yz-plane is a part of a hyperbola (See Fig. 3 (right)).

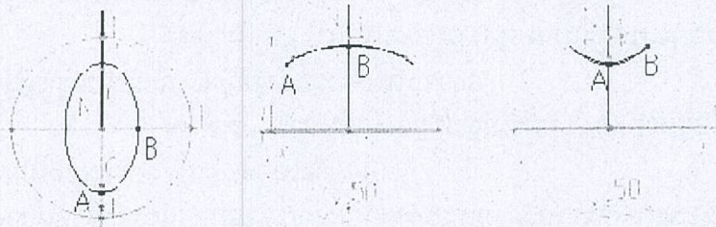


Figure 3: Projections to xy-Plane, xz-Plane, and yz-Planes

## 3. THE FOURTH PARAMETER $d$

Now let us introduce the fourth parameter $d$ as the angle between the asymptote of Equation (5) and z-axis, where the equation of the asymptote is $y = \pm z\tan d$ as in Fig. 4(right). Then Equation (5) is written as a simpler form:

$$-\frac{y^2}{\cos^2 a\tan^2 d} + \frac{z^2}{\cos^2 a} = 1 \quad (z > 0).$$

and this *asymptotic angle* $d$ is defined as

$$\tan d = \tan b/\sin c. \tag{6}$$

Let $D$ be a point on the unit sphere as $D(0, \sin d, \cos d)$(See Fig. 4(right)).

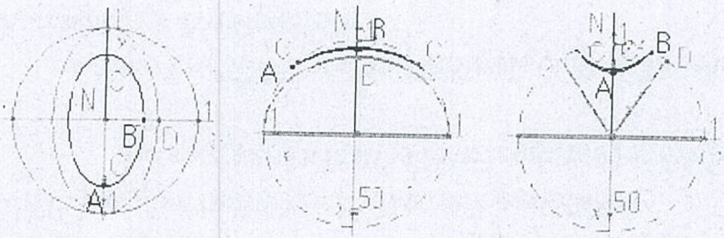

Figure 4: Projections to xy-Plane, xz-Plane, and yz-Plane with Asymptotic Curve

Let us focus on the asymptotic curve which is the intersection of the asymptote and the unit sphere. The projected images of this curve to xy-plane and xz-plane are ellipses(See Fig. 4(left, center)):

$$x^2 + \frac{y^2}{\sin^2 d} = 1,$$
$$x^2 + \frac{z^2}{\cos^2 d} = 1.$$

Moreover. These ellipses look like similar to projected images of spherical ellipses. In fact, it is true.

**Proposition 1.** *The projected images of the asymptotic curve to xy-plane and xz-plane are similar to the projected images of spherical ellipses, respectively.*

*Proof.* It is enough to show that $1 : \sin d = \sin a : \sin b$ and $1 : \cos d = \sin a / \sin c : \cos b$. Using Equation (1) $\cos a = \cos b \cos c$ and Equation (6) $\tan b = \sin c \tan d$,

$$\frac{\sin b}{\sin a} = \frac{\sin b}{\sqrt{1 - \cos^2 b \cos^2 c}} = \frac{\tan b}{\sqrt{1/\cos^2 b - \cos^2 c}} = \frac{\tan b}{\sqrt{\sin^2 c + \tan^2 b}}$$

$$= \frac{\tan d}{\sqrt{1 + \tan^2 d}} = \frac{\sin d}{1}.$$

In the similar way,

$$\frac{\cos d}{1} = \frac{\sin d}{\tan d} = \frac{\sin b / \sin a}{\tan b / \sin c} = \frac{\cos b}{\sin a / \sin c}.$$ □

It is interesting that asymptotic curve looks like outside of spherical ellipse in Fig. 4(left), however, asymptotic curve looks like inside of spherical ellipse in Fig. 4(center).

## 4. SEVEN RELATIONS

Now let us summarize the relations between four parameters $a, b, c$, and $d$. Some of relations are already founded, for example, $\sin b = \sin d \sin a$ and $\cos b \sin c = \cos d \sin a$ in the proof of Proposition 1. In total, there are seven relations.

**Theorem 2.** *There are four relations between three of four parameters:*

$$\begin{aligned} \cos a &= \cos b \cdot \cos c & (7)\\ \tan b &= \sin c \cdot \tan d & (8)\\ \tan c &= \cos d \cdot \tan a & (9)\\ \sin d &\cdot \sin a = \sin b. & (10) \end{aligned}$$

*In addition, there are three relations between four parameters:*

$$\begin{aligned} \cos b &\cdot \sin c = \cos d \cdot \sin a & (11)\\ \tan b &= \cos c \cdot \sin d \cdot \tan a & (12)\\ \sin b &= \tan c \cdot \tan d \cdot \cos a. & (13) \end{aligned}$$

*Proof.* Equation (7) is the very Equation (1) which is the Pythagorean theorem in spherical geometry. Equation (8) is the very Equation (6) which is the definition of the fourth parameter $d$. Equation (10) is showed in the proof of Proposition 1. Equation (9) is derived from Equations (7), (8), and (10):

$$\tan c = \frac{\sin c}{\cos c} = \frac{\tan b / \tan d}{\cos a / \cos b} = \frac{\sin b}{\tan d \cos a} = \frac{\sin d \sin a}{\tan d \cos a} = \cos d \tan a.$$

Equation (11) is also showed in the proof of Proposition 1. Equation (12) is derived from Equations (7) and (8):

$$\tan b = \frac{\sin b}{\cos b} = \frac{\sin d \sin a}{\cos a / \cos c} = \cos c \sin d \tan a.$$

In the similar way, Equation (13) is derived from Equations (7) and (10):

$$\sin b = \tan b \cos b = \sin c \tan d \frac{\cos a}{\cos c} = \tan c \tan d \cos a. \qquad \square$$

It is amazing that all combinations of trigonometric functions(sin, cos, tan) and four parameters($a, b, c, d$) exist once in Equations (7), (8), (9), and (10). In the same way, all combinations exist once in Equations (11), (12), and (13).

## 5. DUALITIES

To understand the previous seven relations, let us consider the polar spherical ellipse of the spherical ellipse.

**Theorem 3.** *Under the interpretations as follows:*

$$a \leftarrow b, \quad c \leftarrow d, \quad \sin \leftarrow \cos, \quad \tan \leftarrow \cot,$$

*Equations (7) and (10) are dual, and also, Equations (8) and (9) are dual. Furthermore, every equation of Equations (11), (12), and (13) is self-dual.*

*Proof.* It is easy to check these duality in each equation. To understand the reason, we have to consider the polar shperical ellipse. The polar spherical ellipse is the intersection of the polar cone of the cone(Equation (2)) and the unit sphere. The polar cone is the set of normal line of a plane which is tangent to the cone. Equation (2) of the cone $C_0$ is written in the quadratic form:

$$(x \quad y \quad z).\begin{pmatrix} 1/\tan^2 a & 0 & 0 \\ 0 & 1/\tan^2 b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

The tangent plane at $(x_0, y_0, z_0) \in C_0$ is

$$(x_0 \quad y_0 \quad z_0)\begin{pmatrix} 1/\tan^2 a & 0 & 0 \\ 0 & 1/\tan^2 b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

hence a normal of this plane is $(x_1, y_1, z_1) = (x_0/\tan^2 a, y_0/\tan^2 b, z_0)$. $(x_0, y_0, z_0) = (x_1 \tan^2 a, y_1 \tan^2 b, z_1)$ is in $C_0$, that is,

$$(x_1 \tan^2 a \quad y_1 \tan^2 b \quad z_1)\begin{pmatrix} 1/\tan^2 a & 0 & 0 \\ 0 & 1/\tan^2 b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \tan^2 a \\ y_1 \tan^2 b \\ z_1 \end{pmatrix} = 0,$$

$$(x_1 \quad y_1 \quad z_1)\begin{pmatrix} \tan^2 a & 0 & 0 \\ 0 & \tan^2 b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = 0.$$

Therefore, the equation of the polar cone is

$$\frac{x^2}{\cot^2 a} + \frac{y^2}{\cot^2 b} = z^2.$$

Let $(a', b', c', d')$ be the parameters of this polar cone. Since $\cot b > \cot a$, $\tan a' = \cot b$ and $\tan b' = \cot a$, that is, $a' + b = b' + a = \pi/2$. Then,

$$\cos c' = \frac{\cos a'}{\cos b'} = \frac{\sin b}{\sin a} = \sin d.$$

In the same way,

$$\sin d' = \frac{\sin b'}{\sin a'} = \frac{\cos a}{\cos b} = \cos c.$$

These equations implies that $c' + d = d' + c = \pi/2$. In this way, the polar of the polar cone is the cone, and this is the reason why these seven relations have the dual property.

□

## References

[1] Berger, M.: Geometry II Springer-Verlag, Berlin Heidelberg. (1987)

[2] Jennings, G.: Modern Geometry with Applications. Springer-Verlag, New York. (1994)