SOME NEW ELEMENTS OF THE ELLIPTIC BRUNN-MINKOWSKI THEORY

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Abstract

In this paper we present various matrix analogs of notions and inequalities in convex geometry. We employ the well known notion of mixed determinant – an analog of the notion of mixed volume in convex geometry, and introduce the matrix version of Blaschke summation – an analog of the notion of Blaschke summation for convex bodies. With these notions we then can develop some matrix analogs of the convex geometry. In this paper we also present one new inequality analog – the matrix version of Kneser-Süss inequality.

KEYWORDS : Minkowski inequality, Brunn-Minkowski inequality, Kneser-Süss inequality, Minkowski's determinant inequality, Blaschke summation, Matrix Blaschke summation, Mixed determinant, Matrix Kneser-Süss inequality.

1. Introduction

The Brunn-Minkowski theory is a core part of convex geometry. At its foundation lies the Minkowski addition of convex bodies which led to the definition of mixed volume of convex bodies and, implicitly, to the famous Brunn-Minkowski inequality. The latter dates back to 1887. Since then it has led to various notions and a series of inequalities in convex geometry. Various matrix analogs of these notions and inequalities have been well known for a century and have been widely use in mathematical and engineering applications. Our purpose here is to develop an equivalent series of inequalities for positive definite symmetric matrices. Toward this goal we employ the well known notion of mixed determinant – an analog of the notion of mixed volume in convex geometry, and introduce the matrix version of Blaschke summation – an analog of the notion of Blaschke summation for convex bodies. With these notions we then can develop some matrix analogs of the convex geometry. In this paper we also present one new inequality analog – the matrix version of Kneser-Süss inequality.

We begin by restating the definition of the mixed determinant, $D(A_1, A_2, \cdots, A_n)$, analogous to the mixed volume of convex bodies (see [1] or Lutwak [2] for a detailed discussion) and the cofactor matrix, $\mathcal{C}A$. Here and thereafter, all matrices are $n \times n$. For two square matrices, A and B, we introduce the Blaschke Summation $A + B$. Letting $A \cdot B := \sum_{i,j=1}^{n} a_{ij} b_{ij}$, we use the axiomatic properties of the mixed determinant to show the equivalency of $\mathcal{C}A \cdot B$ and n times a mixed determinant $nD(\underbrace{A, \cdots, A}_{n-1}, B)$, and to prove $\mathcal{C}A \cdot B = \lim_{\varepsilon \to 0} \frac{D(A + \varepsilon B) - D(A)}{\varepsilon}$ (Theorem 6) and $\frac{1}{n-1} \mathcal{C}B \cdot A = \lim_{\varepsilon \to 0} \frac{D(A + \varepsilon \cdot B) - D(A)}{\varepsilon}$ (Theorem 7). Subsection 2.3 focuses on three equivalent matrix inequalities. The first two were known before in matrix theory, but the last one is a new addition. All these three inequalities have their analogs in convex geometry and we refer to the Appendix A for their statements.

2. Materials and Methods

2.1. Mixed Determinant and Cofactors.

A well known matrix analog of the convex geometry notion of mixed volume is called mixed determinant. Its definition is restated here as follows:

Definition 1 (Mixed Determinant[1]). *Let $A_1, \cdots, A_r$ be $n \times n$ symmetric matrices, $\lambda_1, \cdots, \lambda_r$ be positive scalars. Then the determinant of $\lambda_1 A_1 + \cdots + \lambda_r A_r$ can be written as*

$$D(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum \lambda_{i_1} \cdots \lambda_{i_n} D(A_{i_1}, \cdots, A_{i_n}),$$

where the sum is taken over all n-tuples of positive integers $(i_1, \cdots, i_n)$ whose entries do not exceed r. The coefficient $D(A_{i_1}, \cdots, A_{i_n})$. with A_{i_k}, $1 \le k \le n$ from the set $\{A_1, \cdots, A_r\}$, is called the **mixed determinant** *of the matrices $A_{i_1}, \cdots, A_{i_n}$*

Properties of Mixed Determinants: Let $A_1, \cdots, A_n, A, B$ and B' be $n \times n$ matrices, $\lambda_1, \cdots, \lambda_n$ be positive scalars.

1.
$$\begin{aligned} D(\underbrace{A, \cdots, A}_{n-1}, B) &= D(\underbrace{A, \cdots, A}_{n-2}, B, A) \\ &= \cdots \\ &= D(A, B, \underbrace{A, \cdots, A}_{n-2}) \\ &= D(B, \underbrace{A, \cdots, A}_{n-1}) \end{aligned}$$

[1]The author choose to restate this definition of mixed determinant in a way analogous to the definition of mixed volume in convex geometry [1], [2].

In fact, the mixed determinant is symmetric in its arguments, so in a larger generality one has:

(1) $$D(\underbrace{A,\cdots,A}_{n-k},\underbrace{B,\cdots,B}_{k}) = \cdots = D(\underbrace{B,\cdots,B}_{k},\underbrace{A,\cdots,A}_{n-k}).$$

We use the notation $D(A,n-k;B,k)$ to represent any of $D(\underbrace{A,\cdots,A}_{n-k},\underbrace{B,\cdots,B}_{k}),\cdots,$ $D(\underbrace{B,\cdots,B}_{k},\underbrace{A,\cdots,A}_{n-k})$ in (1).

2.

(2) $$D(\lambda_1 A_1,\cdots,\lambda_n A_n) = \lambda_1\cdots\lambda_n D(A_1,\cdots,A_n).$$

3.

(3) $$D(A_1,\cdots,A_{n-1},B+B') = D(A_1,\cdots,A_{n-1},B) + D(A,\cdots,A_{n-1},B').$$

In particular,

$$D(\underbrace{A,\cdots,A}_{n-1},B+B') = D(\underbrace{A,\cdots,A}_{n-1},B) + D(\underbrace{A,\cdots,A}_{n-1},B').$$

The properties in (2) and (3) follow from the n-**linearity** of the mixed determinant.

One can show that for $n\times n$ matrices A and B:

(4) $$D(\underbrace{A,\cdots,A}_{n-1},B) = \frac{1}{n}\left(\begin{vmatrix} a_1\\ \vdots\\ a_{n-1}\\ b_n\end{vmatrix} + \cdots + \begin{vmatrix} b_1\\ a_2\\ \vdots\\ a_n\end{vmatrix}\right),$$

of which the generalization gives an alternative definition of the mixed determinant as in the following remark:

Remark 2. *A mixed determinant $D(A_1,A_2,\cdots,A_n)$ of $n\times n$ matrices $A_1,A_2,\cdots,A_n$ can be regarded as the arithmetic mean of the determinants of all possible matrices which have exactly one row from the corresponding rows of $A_1,A_2,\cdots,A_n$.*

Definition 3 (Cofactor Matrix). *The cofactor matrix, $\mathcal{C}A$, of an $n\times n$ matrix A, is the transpose of the well known classical adjoint of A, thus it is defined by*

(5) $$(\mathcal{C}A)_{ij} := (-1)^{i+j}D(A(i|j))$$

where $A(i|j)$ denotes the $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and the j-th column of the matrix A.

We use a similar notation in matrix theory to represent an analog of the mixed volume $V_1(K,L)$, where K and L are convex bodies, as follows:

Definition 4. *$D_1(A,B)$ is the following mixed determinant of $n \times n$ matrices A and B:*

$$D_1(A,B) := D(\underbrace{A,\cdots,A}_{n-1},B) \tag{6}$$

2.2. Blaschke Summation and Two Analogs of Mixed Volume between Two Convex Bodies. Learning the properties of the notion of Blaschke summation of convex bodies in convex geometry, we introduce its analog in matrix theory as follows:

Definition 5 (Blaschke Summation).

The Blaschke Summation of the $n \times n$ matrices A and B, denoted by $A+B$, is defined as the matrix whose cofactor matrix is the sum of the cofactor matrices of A and B; that is, it satisfies the following equality:

$$\mathcal{C}(A+B) = \mathcal{C}A + \mathcal{C}B. \tag{7}$$

Theorem 6. *Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$. If*

$$A \cdot B := \sum_{i,j} a_{ij} b_{ij},$$

then, for any positive scalar ε,

$$nD_1(A,B) = \mathcal{C}A \cdot B = \lim_{\varepsilon\to 0} \frac{D(A+\varepsilon B) - D(A)}{\varepsilon} \tag{8}$$

Proof *Let $(\mathcal{C}A)_{ij}$ be the (ij)-th entry of $\mathcal{C}A$. Then*

$$\mathcal{C}A \cdot B = \sum_{i,j} (\mathcal{C}A)_{ij} b_{ij} = \begin{vmatrix} a_1 \\ \vdots \\ a_{n-1} \\ b_n \end{vmatrix} + \cdots + \begin{vmatrix} b_1 \\ a_2 \\ \vdots \\ a_n \end{vmatrix},$$

or, using (4), we have

$$\mathcal{C}A \cdot B = nD(\underbrace{A,\cdots,A}_{n-1},B). \tag{9}$$

For A and B of dimension $n \times n$, and a positive scalar ε, we have

$$D(A+\varepsilon B) = \sum_{i=0}^{n} \binom{n}{i} \varepsilon^i D(A, n-i; B, i),$$

where $D(A, n-i; B, i)$ *represents* $D(\underbrace{A, \cdots, A}_{n-i}; \underbrace{B, \cdots, B}_{i})$. *Thus*

$$nD(\underbrace{A, \cdots, A}_{n-1}, B) = \lim_{\varepsilon \to 0} \frac{D(A + \varepsilon B) - D(A)}{\varepsilon},$$

or, by (9), *we obtain*

$$\mathcal{C}A \cdot B = \lim_{\varepsilon \to 0} \frac{D(A + \varepsilon B) - D(A)}{\varepsilon}$$

□

It is natural to regard the product $\mathcal{C}A \cdot B$ as an equivalent of the mixed volume between two convex bodies A and B. The previous theorem was proved by the asymptotic expansion of the determinant of $A + \varepsilon B$ which is similar to the Steiner's polynomial for the volume of $A + \varepsilon B$, where A and B are convex bodies. Another equivalent of the mixed volume between two convex bodies A and B is provided by the next result.

Theorem 7. *Let* A, B *be* $n \times n$ *positive definite symmetric matrices,* ε *be a positive scalar. Then*

$$\frac{1}{n-1} \mathcal{C}B \cdot A = \lim_{\varepsilon \to 0} \frac{D(A + \varepsilon \cdot B) - D(A)}{\varepsilon}, \tag{10}$$

where $\varepsilon \cdot B = \varepsilon^{1/(n-1)} B$. □

Here we have symmetry up to a constant. This theorem can be proved in a similar way using also the definition of Blaschke summation and axiomatic properties of the mixed determinant.

2.3. The Matrix Analogs of the Brunn-Minkowski, the Minkowski, the Kneser-Süss Inequalities. The following theorem is a well known inequality proved by Minkowski.

Theorem 8 (Minkowski, "The Brunn-Minkowski inequality" [3], [4], [5], [6]). *Let* A, B *be* $n \times n$ *positive definite symmetric matrices. Then*

$$D(A + B)^{1/n} \geq D(A)^{1/n} + D(B)^{1/n}, \tag{11}$$

with equality if and only if $A = cB$. □

It is called Minkowski's determinant inequality [3], [4], [6], and is a matrix analog of the Brunn-Minkowski inequality in convex geometry. And here are couple of others.

Theorem 9 ("The Minkowski inequality" [2]). *Let* A, B *be* $n \times n$ *positive definite symmetric matrices. Then*

$$D_1(A, B) \geq D(A)^{\frac{n-1}{n}} D(B)^{\frac{1}{n}}, \tag{12}$$

with equality if and only if $A = cB$.

[2]Despite lacking of reference literature, the author believes that this theorem is a well known theorem in matrix theory.

Proof *Using AM-GM inequality:* $\frac{1}{n}\operatorname{tr} Q \geq D(Q)^{1/n}$ *for any positive definite matrix* Q, *and* $A \cdot B := \sum_{i,j} a_{ij}b_{ij}$, *it can be easily proved that for* $n \times n$ *positive definite symmetric matrices* A *and* B,

$$\operatorname{tr}(AB) = A \cdot B \geq nD(A)^{1/n}D(B)^{1/n}, \tag{13}$$

and the equality holds if and only if $AB = cI$, *or* A *is a multiple of* B^{-1}; *that is,* $A = cB^{-1}$. *Then it follows directly from (13) that*

$$\begin{aligned} \mathcal{C}A \cdot B &\geq nD(\mathcal{C}A)^{\frac{1}{n}}D(B)^{\frac{1}{n}} \\ &= nD(A)^{\frac{n-1}{n}}D(B)^{\frac{1}{n}}, \end{aligned}$$

and equality holds if and only if $c_1A^{-1} = \mathcal{C}A = c_2B^{-1}$ *or* $A = cB$, *where* c_1, c_2, c *are constant.* □

This inequality is a matrix version of the Minkowski inequality in convex geometry. It can also be shown that the analog of the Brunn-Minkowski inequality (11) is equivalent to the analog of the Minkowski inequality (12). First we shows (12) implies (11). For any positive definite symmetric matrix Q, it follows from (12) that

$$\begin{aligned} \mathcal{C}Q \cdot Q &= nD(Q)^{(n-1)/n}D(Q)^{1/n} \\ &= nD(\mathcal{C}Q)^{1/n}D(Q)^{1/n} \end{aligned} \tag{14}$$

Letting $Q = A + B$, where A, B are positive definite symmetric matrices, we have

$$\begin{aligned} D(A+B)^{1/n} &= \frac{\mathcal{C}Q \cdot (A+B)}{nD(\mathcal{C}Q)^{1/n}} \\ &= \frac{\mathcal{C}Q \cdot A}{nD(\mathcal{C}Q)^{1/n}} + \frac{\mathcal{C}Q \cdot B}{nD(\mathcal{C}Q)^{1/n}} \\ &= \frac{\mathcal{C}Q \cdot A}{nD(Q)^{\frac{n-1}{n}}} + \frac{\mathcal{C}Q \cdot B}{nD(Q)^{\frac{n-1}{n}}} \\ &\geq D(A)^{1/n} + D(B)^{1/n} \end{aligned}$$

The last inequality follows from (12). This concludes that (12) implies (11). We will now show (11) implies (12). By (11) and with ε being a positive scalar, we have

$$\begin{aligned} &\frac{D(A+\varepsilon B) - D(A)}{\varepsilon} \\ &= \frac{\left(D(A+\varepsilon B)^{1/n}\right)^n - D(A)}{\varepsilon} \\ &\geq \frac{\left(D(A)^{1/n} + D(\varepsilon B)^{1/n}\right)^n - D(A)}{\varepsilon} \\ &= \frac{\left(D(A)^{1/n} + \varepsilon D(B)^{1/n}\right)^n - D(A)}{\varepsilon} \\ &= \frac{\left(D(A) + \binom{n}{1}D(A)^{(n-1)/n}\varepsilon D(B)^{1/n} + \binom{n}{2}D(A)^{(n-2)/n}\varepsilon^2 D(B)^{2/n} + \cdots\right) - D(A)}{\varepsilon} \end{aligned}$$

and as ε approaches 0, we infer that

$$\lim_{\varepsilon \to 0} \frac{D(A+\varepsilon B) - D(A)}{\varepsilon} \geq \binom{n}{1} D(A)^{\frac{n-1}{n}} D(B)^{\frac{1}{n}},$$

which is

$$\mathcal{C}A \cdot B \geq nD(A)^{\frac{n-1}{n}} D(B)^{\frac{1}{n}}$$

or

$$D_1(A, B) \geq D(A)^{\frac{n-1}{n}} D(B)^{\frac{1}{n}}.$$

This concludes the proof that (11) implies (12).

Theorem 10 ("The Kneser-Süss inequality"). *Let A, B be $n \times n$ positive definite symmetric matrices. Then*

$$D(A+B)^{\frac{n-1}{n}} \geq D(A)^{\frac{n-1}{n}} + D(B)^{\frac{n-1}{n}}, \tag{15}$$

with equality if and only if $A = cB$.

Proof *To prove this matrix version of Knesser-Süss inequality, it suffices to show that it is equivalent to the analog of the Brunn-Minkowski inequality (11). Using (11) we have*

$$\begin{aligned} D(A+B)^{\frac{n-1}{n}} &= D(\mathcal{C}A + \mathcal{C}B)^{1/n} \\ &\geq D(\mathcal{C}A)^{1/n} + D(\mathcal{C}B)^{1/n} \\ &= D(A)^{\frac{n-1}{n}} + D(B)^{\frac{n-1}{n}} \end{aligned} \tag{16}$$

This shows that (11) implies (15).

It can be easily verified that an $n \times n$ matrix A is positive definite symmetric if and only if its cofactor matrix $\mathcal{C}A$ is a positive definite symmetric. Let $X = \mathcal{C}A$, $Y = \mathcal{C}B$. Since A and B are positive definite symmetric then so are X and Y. Using the definition of Blaschke addition and (15), we obtain

$$\begin{aligned} D(X+Y)^{1/n} &= D(\mathcal{C}A + \mathcal{C}B)^{1/n} \\ &= D(\mathcal{C}(A+B))^{1/n} \\ &= D(A+B)^{\frac{n-1}{n}} \\ &\geq D(A)^{\frac{n-1}{n}} + D(B)^{\frac{n-1}{n}} \\ &= D(\mathcal{C}A)^{1/n} + D(\mathcal{C}B)^{1/n} \\ &= D(X)^{1/n} + D(Y)^{1/n}. \end{aligned} \tag{17}$$

This shows that (15) implies (11), and the theorem is proved. □

The last inequality was unknown in matrix theory. One may recognize the equivalent of this inequality in convex geometry, where volumes replace the determinants and convex bodies replace matrices. The convexity version of the last two theorems are given in Appendix A.

3. Conclusions

The matrix version of Blaschke summation and the AM-GM inequality, $\frac{1}{n}\operatorname{tr} Q \geq D(Q)^{1/n}$ as in the proof of Theorem 9, play important roles in the derivation of matrix analogs of notions and inequalities in convex geometry. These analogs look very similar to their convex geometry version ones. The author believes that a plethora of other matrix inequalities can be obtained by choosing strategic positive definite matrices Q in the AM-GM inequality.

4. Acknowledgements

I wish to thank the Department of Mathematics, Polytechnic University, where half of this work was conducted, especially Professor Erwin Lutwak, Head of the department and my former Ph.D. advisor, Associate Professor Chandni Shah, and Dr. Alina Stancu. A special thank goes to Professor Lutwak for introducing this charming subject of convex geometry and offering me a visiting research/teaching position there at Polytechnic University. I would like to thank Chandni and Alina for all the help and support in practically everything for the pass four years at Polytechnic University, and for proofreading the manuscript of this paper.

Thanks the Department of Control Engineering, Faculty of Engineering, KMITL, where the second half of this work and most of its writing were done. Thanks Associate Professor Dr. Vanchai Riewruja, Head of the department, and Associate Professor Dr. Tawil Paungma, Dean of the faculty, for their kindness in letting me take a leave of absence to go do my research at Polytechnic University.

Appendix A. The Brunn-Minkowski Inequality, the Minkowski Inequality and the Kneser-Süss Inequality in Convex Geometry

Theorem 11 (The Brunn-Minkowski inequality [1], [2], [6]). *Let K, L be convex bodies in $\mathbb{R}^n$. Then*

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{18}$$

with equality if and only if K and L are homothetic. □

The theorem now named after Brunn and Minkowski was discovered (for dimensions ≤ 3) by Brunn (1887, 1889) [7], [8]. Its importance was recognized by Minkowski, who gave an analytic proof for the n-dimensional case (Minkowski 1910 [9]) and characterized the equality case; for the latter, see also Brunn (1894) [10].

Theorem 12 (The Minkowski inequality [1], [2], [6]). *Let K, L be convex bodies in $\mathbb{R}^n$. Then*

$$V_1(K,L) \geq V(K)^{\frac{n-1}{n}} V(L)^{\frac{1}{n}}, \tag{19}$$

with equality if and only if K and L are homothetic. □

Theorem 13 (The Kneser-Süss inequality [2]). *Let K, L be convex bodies in $\mathbb{R}^n$. Then*

$$V(K+L)^{\frac{n-1}{n}} \geq V(K)^{\frac{n-1}{n}} + V(L)^{\frac{n-1}{n}}, \tag{20}$$

with equality if and only if K and L are homothetic. □

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