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Remarks on Weierstrass 6-semigroups

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Abstract. A *numerical semigroup* means a subsemigroup of the additive semigroup $\mathbb{N}_0$ consisting of non-negative integers such that its complement in $\mathbb{N}_0$ is finite. A numerical semigroup H is called an *n-semigroup* if the minimum positive integer in H is n. A numerical semigroup is said to be *Weierstrass* if it is the set $H(P)$ which consists of pole orders at P of regular functions on $C\backslash\{P\}$ for some pointed non-singular curve (C, P). This paper is devoted to the study of Weierstrass 6-semigroups H. Especially we give a Weierstrass 6-semigroup which is not the set $H(P)$ for any ramification point P over a double covering of a non-singular curve.

Key words: Weierstrass semigroup of a point, Cyclic Covering of the projective line, Double covering of a curve, Affine toric variety

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1 Introduction.

Let C be a complete nonsingular irreducible algebraic curve of genus $g \geq 2$ over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. Let $\mathbb{K}(C)$ be the field of rational functions on C. For a point P of C, we set

$$H(P) := \{\alpha \in \mathbb{N}_0 | \text{ there exists } f \in \mathbb{K}(C) \text{ with } (f)_\infty = \alpha P\},$$

which is called the *Weierstrass semigroup of the point P*. We note that $H(P)$ is a numerical semigroup. Hurwitz' question in [3] was whether every numerical semigroup H is Weierstrass. It had been a long-standing problem. Buchweitz [1] finally showed that not every numerical semigroup is Weierstrass. On the other hand, there are a lot of positive results. We are interested in

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the following result: For any $n \leq 5$ every n-semigroup is Weiersrass (See Maclachlan [10], Komeda [6]. Komeda [7] for $n = 3, 4, 5$ respectively).

We are interested in Weierstrass 6-semigroups. In Section 2 we review the results on Weierstrass 6-semigroups in Komeda-Ohbuchi [9]. One of these result is related to the Weierstrass semigroup of a totally ramification point on a cyclic covering of the projective line with degree 6. Another one concerns ramification points on a double covering of a trigonal curve. In Section 3 we study 6-semigroups whose monomial curves are specializations of some affine toric varieties. Such a numerical semigroup is said to be *of toric type*. We know that every semigroup of toric type is Weierstrass by Komeda [6]. We give semigroups of toric type which are not the Weierstrass semigroups of a ramification point on a double covering of a curve.

2 The Weierstrass 6-semigroup of a ramification point on a covering curve

An n-semigroup H is said to be *cyclic* if it is the Weierstrass semigroup of a totally ramification point on a cyclic covering of the projective line with degree n. If n is prime, Kim-Komeda [5] gives a computable necessary and sufficient condition for an n-semigroup to be cyclic. In this section we survey the results on 6-semigroups in Komeda-Ohbuchi [9]. To state them we need a new notion about a numerical semigroup. For an n-semigroup H we set

$$S(H) = \{n, s_1, \ldots, s_{n-1}\}$$

where $s_i = \mathrm{Min}\{h \in H | h \equiv i \bmod n\}$ for each i. We call $S(H)$ the *standard basis* for H.

Theorem 2.1. *Let H be a 6-semigroup with*

$$S(H) = \{6, 6m_1 + 1, 6m_2 + 2, 6m_3 + 3, 6m_4 + 4, 6m_5 + 5\}.$$

Then it is cyclic if and only if we have

$$m_2 + m_5 \geqq m_3 + m_4,\ m_1 + m_5 \geqq m_2 + m_4 \text{ and } m_1 + m_4 \geqq m_2 + m_3.$$

Moreover, Komeda-Ohbuchi [9] treats the Weierstrass semigroup of a ramification point $\tilde{P}$ over a double covering $f : \tilde{C} \longrightarrow C$. In fact, the paper [9] shows the following:

Theorem 2.2. *Let H be a Weierstrass semigroup of genus $r \geqq 0$, i.e., there exists a pointed curve (C, P) such that $H(P) = H$. For any odd $n \geqq 2c(H)+1$ we set $H_n = 2H + n\mathbb{N}_0$ where $c(H) = \mathrm{Min}\{c \in \mathbb{N}_0 | c + \mathbb{N}_0 \subseteq H\}$. Then there exists a double covering $f : \tilde{C} \longrightarrow C$ with a ramification point $\tilde{P}$ over P such that $H(\tilde{P}) = H_n$.*

We note that every 3-semigroup is Weierstrass. Hence, we get the following:

Corollary 2.3. *Let H be a 3-semigroup with $S(H) = \{3, 3l_1 + 1, 3l_2 + 2\}$ and n an odd integer larger than $2c(H)$. We set $H_n = 2H + n\mathbb{N}_0$. Let C be the curve defined by an equation of the form*

$$y^3 = (x - c_{11}) \cdots (x - c_{1\ 2l_2-l_1+1})(x - c_{21})^2 \cdots (x - c_{2\ 2l_1-l_2})^2$$

and $\pi : C \longrightarrow \mathbb{P}^1$ the morphism sending a point Q to $(1 : x(Q))$. We denote by P the point of C such that $\pi(P) = (0 : 1)$. Let $f : \tilde{C} \longrightarrow C$ be a double covering with the ramification point $\tilde{P}$ over P in Theorem 2.2. Then $H_n = H(\tilde{P})$.

Proof. We have $H(P) = H$ (For example, see Kim-Komeda [4]). By Theorem 2.2 we get the result. □

When H is a 3-semigroup, using Theorem 2.1 we give a criterion for the 6-semigroup H_n in Theorem 2.2 to be non-cyclic.

Proposition 2.4. *Let H be a 3-semigroup with $S(H) = \{3, 3l_1 + 1, 3l_2 + 2\}$ and n an odd integer larger than $2c(H)$. We set $H_n = 2H + n\mathbb{N}_0$.*
i) *If $n \equiv 3 \bmod 6$, then the 6-semigroup H_n is cyclic.*
ii) *Let $n \equiv 1 \bmod 6$. If $2l_1 = l_2$, then the 6-semigroup H_n is cyclic. Otherwise, H_n is not cyclic.*
iii) *Let $n \equiv 5 \bmod 6$. If $l_1 = 2l_2 + 1$, then the 6-semigroup H_n is cyclic. Otherwise, H_n is not cyclic.*

3 6-semigroups of toric type.

In this section we are interested in a Weierstrass 6-semigroup which is not the semigroup $H(\tilde{P})$ of a ramification point $\tilde{P}$ over a double covering $f : \tilde{C} \longrightarrow C$. By Lemma 2 in Komeda [8] we see that $H(f(\tilde{P}))$ is a 3-semigroup if and only if $H(\tilde{P})$ is a 6-semigroup.

Lemma 3.1. *Let $f : \tilde{C} \longrightarrow C$ be a double covering with a ramification point $\tilde{P}$ such that $H(\tilde{P})$ is a 6-semigroup. Let $S(H(f(\tilde{P}))) = \{3, a_1, a_2\}$ with $a_i \equiv i \bmod 3$ for each i and $S(H(\tilde{P})) = \{6, b_1, b_2, b_3, b_4, b_5\}$ with $b_j \equiv j \bmod 6$ for each j. We denote by t the number of ramification points of f. Then we have the equality*

$$\left[\frac{b_1}{6}\right] + \left[\frac{b_3}{6}\right] + \left[\frac{b_5}{6}\right] = \left[\frac{a_1}{3}\right] + \left[\frac{a_2}{3}\right] - 1 + \frac{t}{2},$$

where for any real number r we denote by $[r]$ the largest integer which is less than or equal to r.

Proof. By Riemann-Hurwitz formula we get

$$2\sum_{i=1}^{5}\left[\frac{b_i}{6}\right] - 2 = 2\left(2\left(\left[\frac{a_1}{3}\right] + \left[\frac{a_2}{3}\right]\right) - 2\right) + t.$$

By Lemma 2 in Komeda [8] we must have $b_2 = 2a_1$ and $b_4 = 2a_2$. Hence we get the result. □

Using Lemma 3.1 we can find a 6-semigroups which is not the Weierstrass semigroup of a ramification point over a double covering. To give such a 6-semigroup we introduce a new notation. For a numerical semigroup H we denote by $M(H)$ the minimal set of generators for the semigroup H, which is uniquely determined.

Proposition 3.2. *Let H be a 6-semigroup with $S(H) = \{6, b_1, b_2, b_3, b_4, b_5\}$ and $M(H) = \{6, b_1, b_3, b_5\}$ where $b_i \equiv i \bmod 6$. We assume that H satisfies one of the following:*

i) $b_2 = b_3 + b_5,\ b_4 = b_1 + b_3,$

ii) $b_2 = b_3 + b_5,\ b_4 = 2b_5,\ b_1 < 2b_5,$

iii) $b_2 = 2b_1,\ b_4 = b_1 + b_3,\ b_5 < 2b_1,$

iv) $b_2 = 2b_1$, $b_4 = 2b_5$, $b_3 < b_1 + b_5$.

Then there is no ramification point $\tilde{P}$ over a double covering such that $H(\tilde{P}) = H$.

Proof. Assume that there were a ramification point $\tilde{P}$ over a double covering $f : \tilde{C} \longrightarrow C$ such that $H(\tilde{P}) = H$. First, we consider the case i). By Lemma 3.1 we have

$$\left[\frac{b_1}{6}\right] + \left[\frac{b_3}{6}\right] + \left[\frac{b_5}{6}\right] = \left[\frac{b_3 + b_5}{6}\right] + \left[\frac{b_1 + b_3}{6}\right] - 1 + \frac{t}{2},$$

where t is the number of the ramification points of f. This equality implies that

$$0 = \left[\frac{b_3}{6}\right] + \frac{t}{2},$$

which is a contradiction.

Second, we consider the case ii). By Lemma 3.1 we have

$$\left[\frac{b_1}{6}\right] + \left[\frac{b_3}{6}\right] + \left[\frac{b_5}{6}\right] = \left[\frac{b_3 + b_5}{6}\right] + \left[\frac{2b_5}{6}\right] - 1 + \frac{t}{2},$$

which implies that

$$\left[\frac{b_1}{6}\right] = \left[\frac{2b_5}{6}\right] + \frac{t}{2}.$$

Since $b_1 < 2b_5$, the equality is impossible.

Third, we consider the case iii). By Lemma 3.1 we have

$$\left[\frac{b_1}{6}\right] + \left[\frac{b_3}{6}\right] + \left[\frac{b_5}{6}\right] = \left[\frac{2b_1}{6}\right] + \left[\frac{b_1 + b_3}{6}\right] - 1 + \frac{t}{2}$$

In view of $b_5 < 2b_1$ the equality cannot hold.

Forth, we consider the case iv). By Lemma 3.1 we have

$$\left[\frac{b_1}{6}\right] + \left[\frac{b_3}{6}\right] + \left[\frac{b_5}{6}\right] = \left[\frac{2b_1}{6}\right] + \left[\frac{2b_5}{6}\right] - 1 + \frac{t}{2}.$$

In view of $b_3 < b_1 + b_5$ this is a contradiction. □

To find another kind of Weierstrass 6-semigroup H we construct an affine toric variety whose special fiber is the monomial curve Spec $k[H]$ as in the proof of the following theorem. Such a numerical semigroup H is said to be *of toric type.*

Theorem 3.3. *Let H be a 6-semigroup with $M(H) = \{6, b_1, b_3, b_5\}$ where $b_i \equiv i \bmod 6$. Assume that $b_1 + b_3 < 2b_5$ and $b_3 + b_5 < 2b_1$. Then H is of toric type, hence it is Weierstrass.*

Proof. Since $b_1 + b_3 < 2b_5$ and $b_3 + b_5 < 2b_1$, we have $2b_3 < b_1 + b_5$. There is a generating system of relations among $b_0 = 6, b_1, b_3$ and b_5 as follows:

$$\alpha_0 b_0 = \alpha_3 b_3,\ \alpha_1 b_1 = \frac{2b_1 - b_3 - b_5}{6} b_0 + b_3 + b_5,$$

$$\alpha_5 b_5 = \frac{2b_5 - b_1 - b_3}{6} b_0 + b_1 + b_3,\ \frac{b_1 + b_5 - 2b_3}{6} b_0 + 2b_3 = b_1 + b_5,$$

where we set $\alpha_0 = \dfrac{2b_3}{6}$, $\alpha_1 = 2$, $\alpha_3 = 2$ and $\alpha_5 = 2$. Let $\varphi_H : k[X] = k[X_0, X_1, X_3, X_5] \longrightarrow k[H]$ be the k-algebra homomorphism sending X_i to t^{b_i}. We denote by J the ideal generated by

$$X_0^{\alpha_0} - X_3^{\alpha_3},\ X_1^{\alpha_1} - X_0^{\frac{2b_1 - b_3 - b_5}{6}} X_3 X_5,$$

$$X_5^{\alpha_5} - X_0^{\frac{2b_5 - b_1 - b_3}{6}} X_1 X_3,\ X_0^{\frac{b_1 + b_5 - 2b_3}{6}} X_3^2 - X_1 X_5.$$

To show that H is of toric type, first we have to prove that the ideal $I_H = \operatorname{Ker} \varphi_H$ is equal to J. By Herzog [2] we may take as generators for the ideal I_H one of the following types:

(I) $F = X_i^{\nu_i} - X_j^{\mu_j} X_k^{\mu_k} X_l^{\mu_l}$, where i, j, k and l are distinct, and $\nu_i > 0$, $\mu_j > 0$, $\mu_k \geqq 0$, $\mu_l \geqq 0$,

(II) $F = X_i^{\nu_i} X_j^{\nu_j} - X_k^{\mu_k} X_l^{\mu_l}$, where i, j, k and l are distinct, and $\nu_i > 0$, $\nu_j > 0$, $\mu_k > 0$, $\mu_l > 0$.

Let $F \in I_H$ be of type (I). If $\nu_i > \alpha_i$, then we may decrease the weight of F or reduce this case to the case (II) where we define the weight on the polynomial ring $k[X] = k[X_0, X_1, X_3, X_5]$ as follows: the weight of X_i is a_i and for any non-zero $c \in k$ the weight of c is zero. Let $i = 0$ and $\nu_0 = \alpha_0$. Then we may assume that $F = X_3^2 - X_1^{\mu_1} X_5^{\mu_5}$, because $X_0^{\alpha_0} - X_3^2 \in J$. Moreover, since $X_1^2 - X_0^{\frac{2b_1 - b_3 - b_5}{6}} X_3 X_5$ and $X_5^2 - X_0^{\frac{2b_5 - b_1 - b_3}{6}} X_1 X_3$, we may assume that $F = X_3^2 - X_1 X_5$, which are absurd because of $2b_3 < b_1 + b_5$. If $i \neq 0$ and $\nu_i = \alpha_i$, by a similar method to the case $i = 0$ we get a contradiction. Let $F \in I_H$ be of type (II). We may assume that $i = 0$. If $j = 1, 5$, this is absurd.

If $j = 3$, we may decrease the weight of F. Hence, we get $I_H \subsetneqq J$, which implies that $I_H = J$.

We set

$$d_{01} = \frac{2b_3}{6},\ d_{31} = 1,\ d_{11} = 1,\ d_{12} = 1,\ d_{02} = \frac{2b_1 - b_3 - b_5}{6},$$

$$d_{52} = 1,\ d_{32} = 1,\ d_{51} = 1,\ d_{03} = \frac{2b_5 - b_1 - b_3}{6}.$$

The following three relations in the generating system form a fundamental system of relations:

$$d_{01}b_0 = (d_{31} + d_{32})b_3 \tag{1}$$

$$(d_{11} + d_{12})b_1 = d_{02}b_0 + d_{31}b_3 + d_{51}b_5 \tag{2}$$

$$(d_{51} + d_{52})b_5 = d_{03}b_0 + d_{11}b_1 + d_{32}b_3 \tag{3}$$

because the remaining relation in the generating system is described as ${}^t(2) + {}^t(3)$ where ${}^t(n)$ means the equation whose right side (resp. left side) is the left side (resp. right side) of the equation (n). We attach the vectors $\mathbf{a}_1 = \mathbf{e}_1, \mathbf{a}_2 = \mathbf{e}_2, \mathbf{a}_3 = \mathbf{e}_3, \mathbf{b}_4 = \mathbf{e}_4, \mathbf{a}_5 = \mathbf{e}_5$ and $\mathbf{a}_6 = \mathbf{e}_6 \in \mathbb{Z}^6$ to $d_{01}b_0, d_{31}b_3, d_{11}b_1, d_{12}b_1, d_{02}b_0$ and $d_{52}b_5$ respectively, where $\mathbf{e}_i$ denotes the vector whose i-th component is 1 and j-th component is 0 if $j \neq i$. Hence, the vector $\mathbf{a}_7 = (1, -1, 0, 0, 0, 0)$ is attached to $d_{32}b_3$. Moreover, we attach $\mathbf{a}_8 = (0, -1, 1, 1, -1, 0)$ and $\mathbf{a}_9 = (-1, 0, 0, 1, -1, 1)$ to $d_{51}b_5$ and $d_{03}b_0$ respectively. Let S be the subsemigroup of $\mathbb{Z}^6$ generated by $\mathbf{a}_1, \ldots, \mathbf{a}_9$. To check that S is saturated it suffices to show that

$$\sum_{i=1}^{9} \mathbb{R}_+\mathbf{a}_i \cap \mathbb{Z}^6 \subseteq S,$$

where $\mathbb{R}_+$ denotes the set of non-negative real numbers. Let $\mathbf{a} \in \sum_{i=1}^{9} \mathbb{R}_+\mathbf{a}_i \cap \mathbb{Z}^6$. We may assume that $\mathbf{a} = \sum_{i=1}^{9} \lambda_i\mathbf{a}_i \in \mathbb{Z}^6$ with $0 \leqq \lambda_i < 1$, all i. Then we get

$$\mathbf{a} = (\lambda_1 + \lambda_7 - \lambda_9, \lambda_2 - \lambda_7 - \lambda_8, \lambda_3 + \lambda_8, \lambda_4 + \lambda_8 + \lambda_9, \lambda_5 - \lambda_8 - \lambda_9, \lambda_6 + \lambda_9).$$

We denote by μ_i the i-th component of $\mathbf{a}$. It suffices to show that for $\mu_2 = -1$ or $\mu_5 = -1$ the vector $\mathbf{a}$ belongs to S. Let $\mu_2 = -1$ and $\mu_5 = -1$.

Then $\lambda_7 + \lambda_8 = 1 + \lambda_2$ and $\lambda_8 + \lambda_9 = 1 + \lambda_5$. We may assume that $\mathbf{a} = (0, -1, 1, 1, -1, 1) = \mathbf{a}_8 + \mathbf{a}_6 \in S$. Let $\mu_2 = -1$ and $\mu_5 \geqq 0$. We may assume that $\mathbf{a} = (0, -1, 1, 1, 0, 0) = \mathbf{a}_8 + \mathbf{a}_5 \in S$. Let $\mu_2 \geqq 0$ and $\mu_5 = -1$. We may assume that $\mathbf{a} = (0, 0, 1, 1, -1, 1) = \mathbf{a}_9 + \mathbf{a}_1 + \mathbf{a}_3 \in S$. Thus, the semigroup S is saturated, which implies that Spec $k[S]$ is an affine toric variety. Hence H is of toric type. By Komeda [6] the 6-semigroup is Weierstrass. □

By Proposition 3.2 and Theorem 3.3 we can show that there is a Weierstrass 6-semigroups which is not the Weierstrass semigroup of a ramification point over a double covering.

Corollary 3.4. *Let H be a 6-semigroup with $S(H) = \{6, b_1, b_2, b_3, b_4, b_5\}$ and $M(H) = \{6, b_1, b_3, b_5\}$ where $b_i \equiv i \bmod 6$. We assume that H satisfies $b_1 + b_3 < 2b_5$ and $b_3 + b_5 < 2b_1$. Then H is not the Weierstrass semigroup of a ramification point over a double covering, but it is Weierstrass.*

Proof. Since $b_1 + b_3 < 2b_5$ and $b_3 + b_5 < 2b_1$, we have

$$b_2 = b_3 + b_5 \text{ and } b_4 = b_1 + b_3.$$

By Proposition 3.2 and Theorem 3.3 we get the result. □

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