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CLASSES OF LINEAR OPERATORS IN PROBABILISTIC NORMED SPACE

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ABSTRACT

In this paper we will introduce the classes of linear operators in probabilistic normed space, as the set of all Certainly bounded $L_c(V,V')$, D-bounded $L_D(V,V')$, strongly B-bounded $L_B(V,V')$, and strongly $\psi-$bounded $L_\psi(V,V')$, we then prove they are linear space.

KEY WORDS AND PHRASES: Linear operators, Probabilistic Normed Space.
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1. INTRODUCTION AND PRELIMINARIES

In 1942 K. Menger introduced the notion of probabilistic metric space as a natural generalization of the notion of a metric space; specifically, he looked at the distance concept as a probabilistic rather than a deterministic notion. More precisely, instead of associating a number – the distance $d(p,q)$ - for every pair of elements p, q one should associate a distribution function F_{pq} and, for any positive number x, interpret $F_{pq}(x)$ as the probability that the distance from p to q be less than x

In complete analogy with the classical case, we then have the notion of a probabilistic normed space. This was introduced by A. N. Serstnev in 1963 and later improved by C. Alsina, B. Schweizer, and A. Sklar in 1993.

Before we proceed we must state some definitions, known facts, and, technical results to be used in the sequel, the concepts used are those of [3] and [9]: The space of probability distribution functions (briefly, a d.f.) which we will consider are

$$\Delta^+ = \{F : [-\infty, \infty] \to [0,1] \mid F \text{ is left-continuous, non-decreasing,}$$
$$F(-\infty) = 0 \text{ and } F(+\infty) = 1 \}$$

In particular for any $a \geq 0$, ε_a is the d.f defined by

$$\varepsilon_a(x) = \begin{cases} 0, & \text{if } x \leq a, \\ 1, & \text{if } x > a. \end{cases}$$

The space Δ^+ is partially ordered by the usual pointwise ordering of functions, the maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

A *triangle function* is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit, viz. for all $F, G, H \in \Delta^+$, we have

$$\tau(\tau(F,G),H) = \tau(F,\tau(G,H)),$$
$$\tau(F,G) = \tau(G,F),$$
$$\tau(F,H) \leq \tau(G,H) \text{ if } F \leq G,$$
$$\tau(F,\varepsilon_0) = F.$$

Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+. Typical continuous triangle functions are convolution and the operations τ_T and τ_{T*}, which are given by

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s),G(t)),$$

and

$$\tau_{T^*}(F,G)(x) = \inf_{s+t=x} T^*(F(s),G(t)),$$

for all F, G in Δ^+ and all x in $\Re$ [9; Secs.7.2 and 7.3]. Here T is a continuous *t-norm*, i.e. a continuous binary operation on $[0,1]$ that is associative, commuta- tive , nondecreasing and has 1 as identity; T^* is a continuous *t-conorm*, namely a continuous binary operation on $[0,1]$ that is related to continuous t-norm T through

$$T^*(x,y) = 1 - T(1-x, 1-y).$$

The notion of a probabilistic normed space was first introduced by Serstnev in 1963. In 1993, C. Alsina, B. Schweizer and A. Sklar gave a new definition of a probabilistic normed space [2].

DEFINITION 1.1. A *probabilistic normed* space is a quadruple (V,ν,τ,τ^*), where V is a real vector space, τ and τ^* are continuous triangle functions, and ν is a mapping from V into Δ^+ such that, for all p, q in V, the following conditions hold:

(PN1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, θ being the null vector in V;

(PN2) $\nu_{-p} = \nu_p$;

(PN3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;

(PN4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, for all α in [0,1].

If, instead of (PN1), we only have $\nu_\theta = \varepsilon_0$, then we shall speak of a *Probabilistic Pseudo Normed Space*, briefly a PPN space. If the inequality (PN4) is replaced by the equality $\nu_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, then the PN space is called a *Serstnev space*. The pair (V,ν) is said to be a *Probabilistic Seminormed Space* (briefly a PSN space) if $\nu: V \to \Delta^+$ satisfies (PN1) and (PN2).

There is a $(\varepsilon,\lambda)-$ *topology* in the PN space (V,ν,τ,τ^*) which is generated by the family of neighborhoods, N_p of $p \in V$ in the following way:

$$\mathrm{N}_p(\varepsilon,\lambda)=\{N_p(\varepsilon,\lambda)\}_{\varepsilon>0,\lambda\in(0,1)}\ ,\ N_p(\varepsilon,\lambda)=\{q\in V:\nu_{q-p}(\varepsilon)>1-\lambda\}$$

DEFINITION 1.2. Let (V,ν,τ,τ^*) be a PN space and A be the nonempty subset of V. The *probabilistic radius* of A is the function R_A defined on $\Re^+$ by

$$R_A(x)=\begin{cases} l^-\inf\limits_{p\in A}\nu_p(x), & x\in[0,+\infty);\\ 1, & x\in+\infty.\end{cases}$$

where $l^- f(x)$ denotes the left limit of the function f at the point x.

The definition of bounded sets in a probabilistic normed space was defined in [6].

DEFINITION 1.3. A nonempty set A in a PN space (V,ν,τ,τ^*) is said to be:

(a) *Certainly bounded*, if $R_A(x_0)=1$ for some $x_0\in(0,+\infty)$;

(b) *Perhaps bounded*, if one has $R_A(x)<1$ for every $x\in(0,+\infty)$ and $l^-R_A(+\infty)=1$;

(c) *Perhaps unbounded*, if $R_A(x_0)>0$ for some $x_0\in(0,+\infty)$ and $l^-R_A(+\infty)\in(0,1)$;

(d) *Certainly unbounded*, if $l^-R_A(+\infty)=0$, i.e., if $R_A=\varepsilon_\infty$.

Moreover, A will be said to be *distributionanlly bounded*, or simply *D-bounded* if either (a) or (b) holds, i.e., if $R_A\in D^+$; otherwise, i.e., if $R_A\in\Delta^+\setminus D^+$, A will said to be *D-unbounded*.

The definition of a bounded linear operator in PN space previously studied by B. Lafuerza Guillen, J. A. Rodriguez Lallena and C. Sempi [6], I. Jebril, and R. Ali [3] and I. Jebril, and M. S. Noorani [4].

DEFINITION 1.4. Let (V,ν,τ,τ^*) and (V',μ,σ,σ^*) be PN spaces. A linear map $T:V\to V'$ is said to be *bounded* if it satisfies either one of the following conditions.

(a) *Certainly bounded*: if every certainly bounded set A of the space (V,ν,τ,τ^*) has, as image by T a certainly bounded set TA of the space (V',μ,σ,σ^*), i.e., if there exists

$x_0 \in (0,+\infty)$ such that $\nu_P(x_0) = 1$ for all $p \in A$, then there exists $x_1 \in (0,+\infty)$ such that $\mu_{Tp}(x_1) = 1$ for all $p \in A$.

(b) *D-Bounded*: if it maps every D-bounded set of V into a D-bounded set of V', i.e., if, and only if, it satisfies the implication,

$$\lim_{x\to+\infty} \inf_{p\in A} \nu_p(x) = 1 \Rightarrow \lim_{x\to+\infty} \inf_{p\in A} \nu_{Tp}(x) = 1,$$

for every nonempty subset A of V.

(c) *Strongly B-bounded*: if there exists a constant $k > 0$ such that, for every $p \in V$ and for every $x > 0$, $\mu_{Tp}(x) \geq \nu_p(x/k)$, or equivalently if there exists a constant $h > 0$ such that, for every $p \in V$ and for every $x > 0$,

$$\mu_{Tp}(hx) \geq \nu_p(x).$$

(d) *Strongly* $\Psi-$*bounded*: if there exists a $\Psi : [0,\infty) \to [0,\infty)$ such that $\Psi(x) < x$, $\forall x > 0$ so that the following implication holds for every $p \in V$ for every $x > 0$:

$$\nu_p(x) > 1 - x \Rightarrow \mu_{Tp}(\Psi(x)) > 1 - \Psi(x).$$

We shall also need the following lemma which is due to B. Lafuerza Guillen *et al* [8].

LEMMA 1.5. If $f : (V,\nu,\tau,\tau^*) \to (\Re,\mu,\sigma,\sigma^*)$, α is not a positive integer, and A is D-bounded, there is $n \in Z$, such that $n-1 < \alpha < n$, for every $p \in A$ one has

$$\mu_{\alpha f_p} \geq \mu_{nf_p}.$$

2. CLASSES OF LINEAR OPERATORS IN PN SPACE.

Let (V,ν,τ,τ^*) and (V',μ,σ,σ^*) be PN spaces, and let $L(V,V')$ be the vector space of linear operators $T : V \to V'$.

The following is our definition of some classes in PN space.

DEFINITION 2.1. Let (V,ν,τ,τ^*) and (V',μ,σ,σ^*) be PN spaces. Then

1- $L_c(V,V')$ is called a *Certainly bounded subset,* where

$$L_c(V,V') = \{T : V \to V', T \text{ is a certainly bounded linear operators }\}.$$

2- $L_D(V,V')$ is called a *D- bounded subset*, where

$$L_D(V,V') = \{T : V \to V', T \text{ is a D-bounded linear operators }\}.$$

3- $L_B(V,V')$ is called a *B- bounded subset*, where

$$L_B(V,V') = \{T : V \to V', T \text{ is a strongly B-bounded linear operators }\}.$$

4- $L_\psi(V,V')$ is called a ψ *− bounded subset*, where

$$L_\psi(V,V') = \{T : V \to V', T \text{ is a strongly } \psi - \text{bounded linear operators }\}.$$

In Definition 2.1, we are going to prove that $L_c(V,V')$, $L_D(V,V')$, $L_B(V,V')$ and $L_\psi(V,V')$ are vectors space.

THEOREM 2.2. $L_c(V,V')$ is a vector space.

Proof: Let T and S be two certainly bounded linear operators from (V,ν,τ,τ^*) into (V,μ,σ,σ^*) and let A be certainly bounded subset of V. By Definition 1.4, we note that if there exists $x_0 \in (0,+\infty)$ such that $\nu_P(x_0)=1$ for all $p \in A$, then there exists $x_1 \in (0,+\infty)$ such that $\mu_{fp}(x_1)=1$ for all $p \in A$ and if there exists $x_2 \in (0,+\infty)$ such that $\nu_P(x_2)=1$ for all $p \in A$, then there exists $x_3 \in (0,+\infty)$ such that $\mu_{fp}(x_3)=1$ for all $p \in A$. Also

$$\mu_{f_p+g_p}(x_1+x_3) \geq \sigma(\mu_{f_p}(x_1), \mu_{g_p}(x_3)) \geq 1,$$

hence, $\mu_{f_p+g_p}(x_1+x_3)=1$ and $T+S$ is certainly bounded.

Now let $\alpha \in \Re$ and $T \in L_c(V,V')$. Because of (PN2), it suffices to consider the case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αT is certainly bounded.

Proceeding by induction, assume that $\alpha T \in L_c(V,R)$ for $\alpha = 0,1,2,..,n-1$ with $n \in \mathrm{N}$. Then for every $p \in A$,

$$\mu_{nT_p} \geq \sigma(\mu_{(n-1)T_p}, \mu_{T_p}) \geq 1,$$

so that $\mu_{nT_p} = 1$, i.e. αT is certainly bounded. Therefore αT is certainly bounded for every positive integer n. If α is not a positive integer, there is $n \in Z$, such that $n-1 < \alpha < n$; therefore by Lemma 1.5, every $p \in A$ one has

$$\mu_{\alpha Tp} \geq \mu_{nT_p},$$

and whence

$$\mu_{\alpha Tp} = 1,$$

so that αT is certainly bounded.

THEOREM 2.3. $L_D(V,V')$ is a linear space, where $\sigma(D^+,D^+) \subset D^+$

Proof: Let T and S be two D-bounded linear operators from (V,ν,τ,τ^*) into (V',μ,σ,σ^*). Thus, R'_{TA} and R'_{TA} are in D^+. Since, for every $p \in A$, one has

$$\mu_{Tp+Sp} \geq \sigma(\mu_{T_p}, \mu_{S_p}) \geq \sigma(R'_{TA}, R'_{SA}),$$

which belong to D^+, also $R'_{(T+S)A}$ belong to D^+ and $T+S$ is D-bounded.

Now let $\alpha \in \Re$ and $T \in L_D(V,V')$. Because of (PN2), it suffices to consider case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αT is strongly D-bounded.

Proceeding by induction, assume that $\alpha T \in L_D(V,V')$, i.e. $R'_{\alpha TA} \in D^+$ for $\alpha = 0,1,2,\ldots,n-1$ with $n \in N$. Then, for every $p \in A$,

$$\mu_{nTp} \geq \sigma(\mu_{(n-1)Tp}, \mu_{Tp}),$$

and hence

$$R'_{nTp} \geq \sigma(R'_{(n-1)TA}, R'_{TA}),$$

so that $R'_{nTA} \in D^+$ and nT is D-bounded. Therefore nT is D-bounded for every positive integer n. If α is not a positive integer, there is $n \in Z$, such that $n-1 < \alpha < n$; therefore Lemma 1.5, every $p \in A$ one has $\mu_{\alpha T_p} \geq \mu_{nT_p}$, whence $R_{\alpha A} \geq R_{nTA}$ which means that αT is D-bounded.

THEOREM 2.4. $L_B(V,V')$ is a linear space, where $\sigma = \min$.

Proof: Let T and S be two strongly B-bounded linear operators from (V,ν,τ,τ^*) into (V',μ,σ,σ^*). By Definition 1.4, for every $p \in V$ and $x > 0$, there exist $k_1, k_1 > 0$ such that:

$$\mu_{Tp}(x) \geq \nu_p(\frac{x}{k_1}), \quad (1)$$

and

$$\mu_{Sp}(x) \geq \nu_p(\frac{x}{k_2}), \quad (2)$$

from (1) and (2) we note that:

$$\mu_{(T+S)p}(x) = \mu_{Tp+Sp}(x) \geq \sigma(\mu_{Tp}(\frac{x}{2}), \mu_{Sp}(\frac{x}{2})) \\ \geq \sigma((\nu_{Tp}(\frac{x}{2k_1}), \nu_{Sp}(\frac{x}{2k_2})). \quad (3)$$

Choose $k = \max\{2k_1, 2k_2\} + 1$. Thus, $k \geq 2k_1$ and $k \geq 2k_2$, this implies that

$$\frac{x}{2k_1} \geq \frac{x}{k} \quad \text{and} \quad \frac{x}{2k_2} \geq \frac{x}{k}, \quad \forall x \geq 0.$$

Thus;

$$\nu_p(\frac{x}{2k_1}) \geq \nu_p(\frac{x}{k}) \quad \text{and} \quad \nu_p(\frac{x}{2k_2}) \geq \nu_p(\frac{x}{k}), \quad \forall x \geq 0.$$

Thus $\min\left\{\nu_p(\frac{x}{2k_1}), \nu_p(\frac{x}{2k_2})\right\} \geq \nu_p(\frac{x}{k})$.

Now from (3) we get $\mu_{(T+S)p}(x) \geq \nu_p(\frac{x}{k}), \forall x \geq 0$, so that $T+S$ is strongly B-bounded.

Now let $\alpha \in \Re$ and $T \in L_B(V,V')$. Because of (PN2), it suffices to consider case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αT is strongly B-bounded.

Proceeding by induction, assume that $\alpha T \in L_B(V,V')$ for $\alpha = 0,1,2,3,..,n-1$ with $n(\geq 2) \in N$ from (1) and (2), for every $p \in A$, then,

$$\mu_{nTp}(x) \geq \sigma(\mu_{(n-1)Tp}(\frac{x}{2}), \mu_{Sp}(\frac{x}{2})) \geq \sigma(\nu_{(n-1)p}(\frac{x}{2k_1}), \nu_p(\frac{x}{2k_2})) \\ \geq \sigma(\nu_{(n-1)p}(\frac{x}{k}), \nu_p(\frac{x}{k})) \geq \min(\nu_{(n-1)p}(\frac{x}{k}), \nu_p(\frac{x}{k})) = \nu_{(n-1)p}(\frac{x}{k}) > \nu_{np}(\frac{x}{k}).$$

So that αT is strongly B-bounded. Therefore αT is strongly B-bounded for every positive integer n. If α is not a positive integer, there is $n \in Z$, such that $n-1 < \alpha < n$;

therefore Lemma 1.5, every $p \in A$ one has $\mu_{\alpha T_p} \geq \mu_{nT_p}$, which means that αT is strongly B-bounded. This implies that $T + S$ is strongly B-bounded. Hence $L_B(V,V')$ is a linear space.

THEOREM 2.5. $L_\psi(V,V')$ is a linear space, where $\sigma = \min$.

Proof: Let T and S be two strongly $\psi-$bounded linear operator from (V,ν,τ,τ^*) into (V',μ,σ,σ^*). By Definition 1.3, for every $p \in V$ and $x > 0$, there exist $\psi_1(x) < x$ and $\psi_2(y) < y$, such that:

$$\nu_p(x) > 1 - x \Rightarrow \mu_{T_p}(\psi_1(x)) > 1 - \psi_1(x),$$
$$\nu_p(y) > 1 - y \Rightarrow \mu_{S_p}(\psi_2(y)) > 1 - \psi_2(y),$$

let $\psi(x+y) = \psi_1(x) + \psi_2(y) < x + y$, if $\nu_p(x+y) > 1-(x+y)$ then

$$\begin{aligned}\mu_{(T+S)_p}(\psi(x+y)) &= \mu_{T_p+S_p}(\psi_1(x)+\psi_2(y)) \\ &\geq \sigma(\mu_{T_p}(\psi_1(x),\mu_{S_p}(\psi_2(y))) \\ &\geq \min(\mu_{T_p}(\psi_1(x),\mu_{S_p}(\psi_2(y))) \\ &> 1-\psi(x+y).\end{aligned}$$

So $T + S$ is strongly $\psi-$bounded.

Now let $\alpha \in \Re$ and $T \in L_\psi(V,V')$. Because of (PN2), it suffices to consider case $\alpha \geq 0$. If either $\alpha = 0$ or $\alpha = 1$, then αT is strongly $\psi-$bounded.

Proceeding by induction, assume that $\alpha T \in L_\psi(V,V')$ for $\alpha = 0,1,2,\ldots,n-1$ with $n \in \mathrm{N}$. Then, for every $p \in A$, let $\psi(x+y) > \psi_1(x) + \psi_2(y)$

$$\mu_{nT_p}(\psi(x+y)) \geq \sigma(\mu_{(n-1)T_p}(\Psi_1(x),\Psi_2(y)) > 1 - \Psi(x+y).,$$

So that αT is strongly $\psi-$bounded. Therefore αT is strongly $\psi-$bounded for every positive integer n. If α is not a positive integer, there is $n \in Z$, such that $n-1 < \alpha < n$; therefore Lemma 1.4, every $p \in A$ one has $\mu_{\alpha T_p} \geq \mu_{nT_p}$, which means that αT is strongly $\psi-$bounded.

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