Time-Reversibility of Stationary Markov Chains and Queueing Processes

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Abstract

This paper considers the time-reversibility of stationary Markov chains in discrete-time with general state space and its applications to queueing processes. Ōsawa obtained the necessary and sufficient conditions for time-reversibility of Markov chains with general state space and applied these results to queueing processes. His results are introduced here and some time-reversible processes are studied.

Keywords : Time-reversibility, Markov chain, invariant measure, atom, queue, waiting time, queue-length.

1 Introduction

The time-reversibility of Markov processes with countable states was studied by Kolmogorov [2]. He dealt with time-reversibility of discrete-time and continuous-time Markov processes with countable states and deduced the time-reversibility condition for the process, so called Kolmogorov's criterion. Kelly [1] discussed the time-reversibility for various stochastic phenomena with discrete states, in particular queueing models. On the other hand, the time-reversibility of Markov chains with continuous components was dealt with by Ōsawa[3]. He also applied this property to stochastic models, such as waiting time processes in queues, risk reserve processes, dam contents models auto-regressive processes and so on, see Ōsawa ([3], [4], [5], [6], [7]) and Ōsawa and Shima [8]. In this paper, we review some results obtained in their papers and apply these to some queueing processes.

First we introduce the time-reversibility condition for Markov chains with general state space. This result is applied to Markov chains with atomic states. From the viewpoint of applications to Markovian storage models, the concise criteria of the time-reversibility for such a Markov chain is needed. In particular, we are interested in time-reversibility of queueing processes, especially queue-length processes and waiting-time processes.

In section 2. we show the time-reversibility of Markov chain with general state space. Further, time-reversibility for Markov chains having atomic points in the state space is discussed and the necessary and sufficient conditions for time-reversibility of these processes are obtained in the section. Further, we apply these conditions to some queueing processes. In section 3, time-reversibility for queue-length and waiting-time processes in ordinary queues is investigated. In section 4, time-reversibility for Lindley processes is discussed. The results are applied to waiting-time processes and bulk-arrival queues in sections 5 and 6.

2 Time-reversibility of stationary Markov chains

A stochastic process $\{X_n : n \in Z\}$ is said to be time-reversible if random vectors $(X_{n_1}, X_{n_2}, \cdots, X_{n_k})$ and $(X_{m-n_1}, X_{m-n_2}, \cdots, X_{m-n_k})$ have the same distribution for any m, $n_1 < n_2 < \cdots < n_k \in Z$ where Z is the set of all integers. It is easily seen that a time-reversible process is stationary.

Let $\{X_n : n \in Z\}$ be a stationary Markov chain with general state space S and the transition probabilities $P(x, A)$, $x \in S$ and $A \in \mathcal{F}$, where $(S, \mathcal{F})$ is a measurable space. We here deal with the time-reversibility of $\{X_n\}$.

2.1 Time-reversibility condition

Let $\mathcal{B}(S)$ be the set of real-valued $\mathcal{F}$-measurable bounded functions defined on S, then the following notations are used as usual:

$$\nu f = \int_S \nu(dx) f(x),$$
$$Pf(x) = \int_S P(x, dy) f(y),$$

where ν is a positive measure on $\mathcal{F}$ and $f \in \mathcal{B}(S)$.

Theorem 1 (Ōsawa(1985)) *A stationary Markov chain $\{X_n\}$ is time-reversible if and only if there exists a non-zero bounded measure ν on $(S, \mathcal{F})$ satisfying that*

$$\nu(fPg) = \nu(gPf). \tag{1}$$

for any f, $g \in \mathcal{B}(S)$. Then ν is an invariant measure and $\pi(\cdot) = \nu(\cdot)/\nu(S)$ is the stationary probability for the chain $\{X_n\}$.

Remark 1 We now have some remarks.

(1) Equation (1) is equivalent to

$$\int_A \nu(dx) P(x, B) = \int_B \nu(dx) P(x, A).$$

for any $A, B \in \mathcal{F}$.

(2) We should note that, for a stationary Markov chain $\{X_n\}$, the above condition is equivalent to

$$P[X_n \in A, X_{n+1} \in B] = P[X_m \in B, X_{m+1} \in B].$$

for any $A, B \in \mathcal{F}$.

(3) Consider a Markov chain $\{X_n\}$ with countable state space S and transition probabilities $P_{ij}, i, j \in S$. Then time-reversibility condition is given as follows, i.e., there exists a probability measure π such that

$$\pi_i P_{ij} = \pi_j P_{ji},$$

for any $i, j \in S$.

(4) If the transition probability $P(x, \cdot)$ has the density $p(x, y)$ with respect to ν for each $x \in S$, i.e.,

$$P(x, A) = \int_A p(x, y) \nu(dy), \qquad A \in \mathcal{F},$$

then $\{X_n\}$ is time-reversible if and only if there exists a probability density function $(p.d.f.)$ p such that

$$p(x)p(x, y) = p(y)p(y, x),$$

for any $x, y \in S$. Then $p(x)$ is the stationary $p.d.f.$ for $\{X_n\}$.

2.2 Time-reversibility of Markov chains with atoms

Assume that $\delta \in S$ is an atom for a Markov chain $\{X_n\}$ satisfying

$$P(x, \{\delta\}) > 0 \quad \text{for any } x \in S_\delta \equiv S - \{\delta\}, \{\delta\} \in \mathcal{F}. \tag{2}$$

Further, we assume that

$$\int_{S_\delta} \frac{P(\{\delta\}, dx)}{P(x, \{\delta\})} < \infty.$$

Then a criterion for time-reversibility of this process is obtained as follows.

Theorem 2 (Ōsawa(1985)) *A stationary Markov chain* $\{X_n\}$ *defined as above is time-reversible if and only if the transition probability P satisfies the following condition:*

$$P(fP(gP1_\delta))(\delta) = P(gP(fP1_\delta))(\delta), \tag{3}$$

for any $f, g \in \mathcal{B}(S)$ where 1_δ is the indicator function of $\{\delta\}$. Then the time-reversible chain has the stationary distribution

$$\pi(A) = \pi_\delta \int_A \frac{P(\delta, dx)}{P(x, \{\delta\})}, \quad A \in \mathcal{F},$$

where $\pi_\delta \equiv \pi(\{\delta\})$ is given by

$$\pi_\delta = \left\{ 1 + \int_{S_\delta} \frac{P(\delta, dx)}{P(x, \{\delta\})} \right\}^{-1}$$

From this theorem, for the chain with the transition probability density, the following corollary is immediately deduced.

Corollary 3 (Ōsawa(1985)) *When the Markov chain considered in the above theorem has the transition p.d.f. $p(x, y)$ on S_δ, for each $x \in S$, with respect to a positive measure m, the necessary and sufficient condition for time-reversibility is*

$$p(\delta, x)p(x, y)P(y, \{\delta\}) = p(\delta, y)p(y, x)P(x, \{\delta\}), \quad x, y \in S_\delta. \tag{4}$$

This time-reversible chain has the following stationary distribution:

$$p.d.f. \quad p(x) = \pi_\delta \frac{p(\delta, x)}{P(x, \{\delta\})}, \quad x \in S_\delta,$$

$$\pi_\delta = \left\{ 1 + \int_{S_\delta} m(dx) \frac{p(\delta, x)}{P(x, \{\delta\})} \right\}^{-1}$$

Remark 2 Consider the Markov chain with countable state space and transition probabilities P_{ij}. If there exists a state δ such that

$$P_{j\delta} > 0 \quad \text{for any state } j \in S_\delta,$$

then the necessary and sufficient condition for time-reversibility is

$$P_{\delta i}P_{ij}P_{j\delta} = P_{\delta j}P_{ji}P_{i\delta}, \quad \text{for any } i, j, k \in S_\delta. \tag{5}$$

3 Time-reversibility of some queueing processes

In this section, we consider the time-reversibility of some queue-length processes.

3.1 $M/M/1$ queue

Consider the stationary $M/M/1$ queue with the arrival rate λ and the service rate μ. Let $\{t_n\}$ be a sequence of arrival and departure epochs, $Q(t)$ be the system size at time t and $Q_n = Q(t_n+)$ where t_n+ means immediately after t_n. Then the process $\{Q_n\}$ is a Markov chain with transition probabilities;

$$P_{i,i+1} = \frac{\lambda}{\lambda + \mu} = \frac{\rho}{1 + \rho}, \quad P_{i,i-1} = \frac{\mu}{\lambda + \mu} = \frac{1}{1 + \rho}, \quad i \geq 1,$$
$$P_{01} = 1, \quad P_{ij} = 0, \quad \text{otherwise},$$

where $\rho = \lambda/\mu < 1$. For a measure ν such that $\nu(i) = \rho^i$, we have

$$\nu(i)P_{i,i+1} = \nu(i+1)P_{i+1,i}, \quad i \geq 0.$$

Thus $\{Q_n\}$ is time-reversible with the stationary distribution

$$\pi(i) = (1 - \rho)\rho^i, \quad i \geq 0.$$

Let W_n be the waiting time of the nth customer, excluding his service time, then the process $\{W_n\}$ is a Markov chain with the state space $R_+ = [0, \infty)$, and the following transition p.d.f. $p(x, y)$;

$$p(x, y) = \begin{cases} \dfrac{\lambda\mu}{\lambda + \mu} \exp\{-\lambda(x - y)\}, & x \geq y > 0, \\[2mm] \dfrac{\lambda\mu}{\lambda + \mu} \exp\{-\mu(y - x)\}, & y > x \geq 0, \end{cases}$$

$$P(x, \{0\}) = \frac{\mu}{\lambda + \mu} \exp(-\lambda x), \qquad x \geq 0.$$

Thus state 0 is the only one atom in R_+, and we can easily have

$$p(0, x)p(x, y)P(y, \{0\}) = p(0, y)p(y, x)P(x, \{0\}),$$

for any $x, y \in R_+$. Hence, from Corollary 3, $\{W_n\}$ is time-reversible and has the well-known stationary distribution

$$\pi(dx) = \begin{cases} 1 - \rho, & x = 0, \\ \lambda(1 - \rho)\exp\{-(\mu - \lambda)x\}dx, & x > 0. \end{cases}$$

3.2 Discrete-time queues

In a discrete-time queue, arrivals and departures occur just after and before the integer time point, respectively. Denote the number of arrivals and departures at time $n \in Z$ as A_n and D_n, respectively. Further, denote the queue length at time n as Q_n, then the process $\{Q_n\}$ is formulated by

$$Q_{n+1} = Q_n + A_n - D_{n+1}.$$

Assume that A_n's are muturelly independent and have a common distribution

$$\alpha(k) = \mathbf{P}[A_n = k], \qquad k = 0, 1, 2, \cdots,$$

and D_{n+1} is depend only on the present state $Q_n + A_n$;

$$d(i + j, k) = \mathbf{P}[D_{n+1} = k \mid Q_n = i, A_n = j], \qquad k = 0, 1, 2, \cdots, i + j.$$

Then $\{Q_n\}$ is a Markov chain with transition probabilities;

$$P_{ij} = \sum_{k=\max(j-i,0)}^{\infty} \alpha(k)d(i + k, i + k - j).$$

Consider a discrete-time queue where, at time n, at least one customer arrives and departs with probabilities

$$\alpha(0) = 1 - p, \ \alpha(1) = p, \quad d(i, 0) = 1 - q, \ d(i, 1) = q,$$

for $i \geq 1$ and $0 < p < q < 1$. This queue is denoted by the discrete-time $M/M/1$. For the queue-length process $\{Q_n\}$ in this queue, transition probabilities are given by

$$P_{01} = p(1 - q), \qquad P_{00} = pq + (1 - p),$$
$$P_{i,i+1} = p(1 - q), \quad P_{ii} = pq + (1 - p)(1 - q), \quad P_{i,i-1} = (1 - p)q, \quad i \geq 1.$$

Let ν be a measure such that

$$\nu(i) = \rho^i, \qquad \rho = \frac{(1 - q)p}{(1 - p)q},$$

then we have

$$\nu(i)P_{i,i+1} = \nu(i + 1)P_{i+1,i}, \quad i \geq 0.$$

Thus $\{Q_n\}$ is time-reversible with the stationary distribution

$$\pi(i) = (1 - \rho)\rho^i, \quad i \geq 0.$$

Let W_n be the waiting time of the nth customer, excluding his service time, where we should note that n is different from time n. Then the process $\{W_n\}$ is a Markov chain with the state space $Z_+ = \{i \in Z; i \geq 0\}$, and transition probabilities

$$P_{ij} = \begin{cases} cp(1 - q)^{j-i}, & j \geq i > 0, \\ cp(1 - p)^{i-j}, & i > j \geq 1, \\ c(1 - p)^i, & j = 0, \end{cases}$$

$$P_{0i} = \begin{cases} cp(1 - q)^i, & i > 0, \\ c, & j = 0, \end{cases}$$

where $c = q/(p + q - pq)$.

We can find that the relations

$$P_{0i}P_{ij}P_{j0} = P_{0j}P_{ji}P_{i0},$$

hold for any $i, j \in Z_+$. Hence, from Remark 2, $\{W_n\}$ is time-reversible and has the stationary distribution

$$\pi(i) = \begin{cases} \dfrac{q - p}{(1 - p)q}, & i = 0, \\ \pi(0)p\left(\dfrac{1 - q}{1 - p}\right)^i, & i \geq 1. \end{cases}$$

4 Time-reversibility of Lindley processes

The Lindley process $\{X_n\}$ is defined by

$$X_{n+1} = \max(0, X_n + U_n), \quad n \in Z,$$

where $\{U_n\}$ is a sequence of independent and identically distributed (i.i.d.) random variables with $E[U_n] < 0$. Then $\{X_n\}$ is a Markov chain and has the stationary distribution. This process can be seen in various storage models for inventories, dams, insurance risk and so on. In particular, the waiting-time process in queues is most popular. In this section, we investigate the time-reversibility of stationary Lindley processes and waiting-time processes in queues.

4.1 Lindley processes on real line

Assume that $\{U_n\}$ has a common distribution function (d.f.) $K(x)$ on real line $R = (-\infty, \infty)$. then the Lindley process $\{X_n\}$ is a Markov chain with the state space $R_+ = [0, \infty)$. We also make following assumptions for the d.f. $K(x)$;

(C1) $K(x)$ has a continuous density $k(x)$ and $K(x) > 0$, for all real x.

(C2) $\displaystyle\int_0^\infty \frac{dK(x)}{K(-x)} < \infty$.

(C3) $\displaystyle\int_{-\infty}^\infty x\,dK(x) < 0$.

(C3) indicates $E[U_n] < 0$ and $\{X_n\}$ has the stationary distribution.

Under these conditions, the transition probabilities of $\{X_n\}$ are given as

$$p(x,y) = k(y-x), \quad (x \geq 0, y > 0),$$
$$P(x, \{0\}) = K(-x), \quad (x \geq 0).$$

Corollary 3 implies the time-reversibility condition is

$$k(x)k(y-x)K(-y) = k(y)k(x-y)K(-x), \qquad x,\ y > 0.$$

After some consideration, we can find that the p.d.f. $k(x)$ satisfying this relation is given by

$$k(x) = \begin{cases} \dfrac{ab}{a+b} \exp(ax), & x < 0, \\[3mm] \dfrac{ab}{a+b} \exp(-bx), & x \geq 0. \end{cases}$$

Thus we have the following time-reversibility condition for $\{X_n\}$.

Theorem 4 (Ōsawa(1985)) *The stationary Lindley process under the conditions* (C1) *through* (C3) *is time-reversible if and only if the d.f.* $K(x)$ *is the two-sided exponential distribution.*

4.2 Lindley processes with discrete states

Assume that $\{U_n\}$ has a common distribution $\{u(k)\}$;

$$u(k) = \mathbf{P}[U_n = k], \qquad k \in Z,$$

where Z is the set of all integers, then the Lindley process $\{X_n\}$ is a Markov chain with the state space $Z_+ = \{0, 1, 2, \cdots\}$. For this chain, the transition probabilities are written as, for $i \in Z_+$,

$$P_{ij} = \begin{cases} u(j-i), & j \neq 0; \\[3mm] \displaystyle\sum_{k=-\infty}^{-i} u(k), & j = 0. \end{cases} \tag{6}$$

Theorem 5 (Ōsawa and Shima(1999)) *Suppose that $u(k) > 0$ for all integers k, then it is necessary and sufficient for the Lindley process $\{X_n\}$ with discrete states to be time-reversible that the distribution $\{u(k)\}$ is of the two-sided geometric type given by*

$$u(k) = \begin{cases} c_+ a^{k-1}, & k = 1, 2, \cdots, \\ c_0, & k = 0, \\ c_- b^{-k-1}, & k = -1, -2, \cdots, \end{cases} \tag{7}$$

where constants $0 < a < b < 1$, c_-, c_0 and c_+ satisfy

$$\frac{c_+}{1-a} + c_0 + \frac{c_-}{1-b} = 1,$$

with $c_- a = c_+ b$.

Corollary 6 *Suppose that the distribution $\{u(k)\}$ is of the two-sided geometric type given by (7) with $c_- a = c_+ b$, then the Lindley process $\{X_n\}$ is time-reversible with respect to a measure*

$$\nu(k) = \begin{cases} 1, & k = 0, \\ (1-b)\left(\dfrac{a}{b}\right)^k, & k = 1, 2, \cdots. \end{cases}$$

Moreover, $\{X_n\}$ has the stationary distribution

$$\pi(k) = \begin{cases} \dfrac{b-a}{b(1-a)}, & k = 0, \\ (1-b)\pi(0)\left(\dfrac{a}{b}\right)^k, & k = 1, 2, \cdots. \end{cases}$$

Corollary 7 *Assume that $\{a(k)\}$ is a geometric distribution given by*

$$a(k) = (1-a)a^k, \qquad k = 0, 1, 2, \cdots, \tag{8}$$

for $0 < a < 1$, then $\{u(k)\}$ is the distribution of the form (7) with $c_- p = c_+ q$ if and only if $a = p$ and $\{d(k)\}$ is also a geometric distribution such as

$$d(k) = (1-q)q^k, \qquad k = 0, 1, 2, \cdots. \tag{9}$$

5 Time-reversibility of waiting-time processes

Consider the stationary $GI/GI/1$ queue. Let S_n be the service time of nth customer with the d.f. $B(x)$ and T_n be the interarrival time between nth and $n + 1$st arrival times with d.f. $A(x)$. We assume that $\{S_n\}$ and $\{T_n\}$ are muturelly independent and these distributions have finite means;

$$\lambda^{-1} = \int_0^\infty x\, dA(x), \qquad \mu^{-1} = \int_0^\infty x\, dB(x),$$

where $\rho = \lambda/\mu < 1$.

The waiting-time process $\{W_n\}$ is defined as a Lindley process;

$$W_{n+1} = \max(0, W_n + U_n), \quad U_n \equiv S_n - T_n.$$

Note that $\{U_n\}$ has a common d.f. $K(x)$;

$$K(x) = \int_0^\infty B(x+t)dA(t) = \int_0^\infty \{1 - A(t-x)\}dB(t), \quad -\infty < \dot{x} < \infty.$$

Thus we have the following time-reversibility condition for $\{W_n\}$ by using Theorem 4.

Theorem 8 (Ōsawa(1985)) *For the stationary waiting-time in the $M/GI/1$ and the $GI/M/1$ queues, the time-reversible chain is only for the $M/M/1$.*

In the same way as the ordinary queue in continuous time, the waiting-time process $\{W_n\}$ in the discrete-time queue is defined by

$$W_{n+1} = \max(0, W_n + U_n), \quad U_n \equiv S_n - T_n.$$

Note that $\{U_n\}$ has a common distribution;

$$u(k) = \sum_{j=\max(k+1,1)}^\infty b(j)a(j-k) = \sum_{j=\max(-k+1,1)}^\infty a(j)b(j+k), \quad k \in Z.$$

Theorem 9 (Ōsawa and Shima(1999)) *For waiting-time processes in the discrete-time $M/GI/1$ and $GI/M/1$ queues, the reversible one is only for the $M/M/1$.*

6 An application to a bulk-arrival queue

We now consider the other reversible queueing model. Consider the following bulk-arrival queue $M^A/M/1$. The arrival stream forms a Poisson process with parameter λ and the number of customers who enter the queue upon arrival (the batch size) is a random variable A with the distribution $\{\alpha(k)\}$. The service time of each customer is assumed to be exponential with mean μ^{-1}. Let Q_n be the number of customers in the system immediately before the time t_n at which the n th arrival occurs, A_n be the batch size upon n th arrival and D'_{n+1} be the number of customers who terminate the service and depart from the system during the time interval $[t_n, t_{n+1})$. Then the queue-length process $\{Q_n\}$ is described by

$$Q_{n+1} = Q_n + A_n - D'_{n+1}.$$

Here, D'_{n+1} is dependent only on $X_n + A_n$. That is, let

$$d(i,j) = P[D'_{n+1} = j \mid X_n + A_n = i], \qquad \text{for } 0 \le j \le i,$$

then we have

$$d(i,j) = \begin{cases} \dfrac{\rho}{(1+\rho)^{j+1}}, & j = 0, 1, \cdots i, \\[3mm] \dfrac{1}{(1+\rho)^i}, & j = i, \end{cases}$$

here $\rho = \lambda/\mu$.

Introduce a sequence of independent and identically distributed random variables $\{D_n\}$ with a common distribution

$$d(k) = \frac{\rho}{(1+\rho)^{k+1}}, \qquad k = 0, 1, \cdots,$$

then the queue-length process $\{Q_n\}$ is equivalent to the Lindley process with discrete states. Therefore, from Corollary 7, we have the following.

Theorem 10 For the above bulk-arrival $M^A/M/1$ queue, the queue-length process $\{Q_n\}$ is time-reversible if and only if the batch size A has a geometric distribution with $a(1+\rho) < 1$. Then $\{Q_n\}$ has the stationary distribution

$$\pi(k) = \begin{cases} \dfrac{1 - a(1+\rho)}{1 - a}, & k = 0, \\[3mm] \rho a \{a(1+\rho)\}^{k-1} \pi(0), & k = 1, 2, \cdots. \end{cases}$$

7 Comments

We discussed the time-reversibility with applications to queueing processes by reviewing a series of Ōsawa's works for the time-reversibility. This notion has made theory of queueing networks develop rapidly, see Kelly [1]. However, there are not so many works for the time-reversibility of processes with continuous states and of related problems.

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