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Polystyrene Strength Models

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Abstract

By examining simple models we determine the strength of slabs of polystyrene material manufactured under varying mould fill ratio conditions. We then extend the models to determine the strength of material obtained by expanding mixtures of virgin and recycled polystyrene beads.

I. Introduction

Expanded polystyrene is used extensively as shape-moulded packaging foam; for example for containing fruit, cameras etc. The moulded material consists of expanded beads of polymerised styrene. The material is valued because of its lightness, low cost and reasonable strength.

The raw material is formed by adding a blowing agent (isometric pentane) to polymerised styrene to form spherical beads approximately 1mm in diameter. These beads are then placed in a mould and steam heated. The solid pentane within the beads gasifies causing the beads to expand to about 2.5mm diameter. After cooling a second stage of expansion blows the beads up to about 3.5mm in diameter. The expanded beads compete for space within the mould so that contact surfaces between beads are formed. The mould is then cooled. The beads maintain their shape after cooling because of the solidification of the styrene fibers within the beads, and the beads fuse together at the surfaces of contact. For a more complete description of the process, see [1], [2], [3] and [4].

Significant saving in cost can be made if recycled beads are mixed with virgin beads. The looks and strength of the final product may be compromised by doing this but in certain applications looks are not important and, providing the strength degradation is not too great, the eventual product is acceptable but cheaper than the product obtained using just virgin beads. The recycled beads are simply mixed with the unexpanded beads in the desired proportions and steamed in the mould in the normal way. If the proportion of recycled beads is too great (greater than about 15%), then the product may not be strong enough for use. The determination of the relationship between the strength and the composition of the mixture used is the subject of this article.

This problem arose out of an Mathematics-in-Industry-Study-Group meeting held at QUT in February 1999 moderated by Frank deHoog and Warren Wood. Context details are contained in the Proceedings, see [1]. The present article corrects and extends results reported in the Proceedings.

In the main body of the paper we first describe the strength model used for all subsequent calculations. We then introduce a bubble model and use it to determine the strength of polystyrene material constructed using just virgin beads. Using similar ideas a model of the strength of material made up of mixtures of virgin and recycled beads is developed. In the conclusion we discuss the possible application of this work to problems of general interest in solid mechanics.

2. The Models

2.1 The Strength Model

The fusion between two beads is over a finite surface area and experiments show that the cell structure in the fused region is identical to that within the remaining parts of the beads. Thus one would anticipate that the scaled tensile, compressive or shearing strength of the finished product to be given by

$$S_m = N_{vv}s_{vv} + N_{rr}s_{rr} + N_{vr}s_{vr}, \qquad (1)$$

where N_{vv}, N_{rr}, N_{vr} are the expected number of virgin-virgin, recycle-recycled and virgin-recycle contacts per unit surface area, and s_{vv}, s_{rr}, s_{vr} are scaled contact surface areas associated with the various contact types. The appropriate scaling factor for strength will depend on context. In sections to follow we quantify the terms appearing in the above equation. In applications of interest the product may fail in shear, buckling; or compression, and failure normally results in the rupture of bead to bead contacts. Thus contact surface area effectively determines both the elastic and rupture properties of the final material, with the additional effect of geometry determining the behaviour of manufactured items.

2.2 The Bubble Model: virgin beads

In the absence of a mould to inhibit expansion each bead will expand spherically. In the presence of the mould beads will initially expand spherically until they fill the mould. Further expansion of the beads will be inhibited by the presence of the mould walls; individual beads will compete for available space and contact surfaces will be formed. As a simple model of the process we examine the expansion of a single spherical bead, (a bubble) of initial radius a_0 (volume V_0) contained within a rigid box with n sides, simulating the bead to bead or bead to wall contacts. Observations show that n lies in the range 7 to 9. We make calculations for the regular polyhedra box cases; the cube with 6 square faces (and contacts), the octahedron with 8 triangular faces, and the dodecadron with 12 pentagonal faces. The dodecahedron is the closest packing case. The size of the box is chosen so that the bubble's surface just touches the walls of the box; thus a_0 is the inscribed radius of the polyhedron, which for the cube is half it's length. The two expansion stages are displayed in Figure 1. In the cube case the sides of the box are of length $2a_0$ the volume is $8a_0^3$, and the bubble makes contact at n=6 points. For simplicity of presentation in the text to follow we'll refer specifically to the cube case.

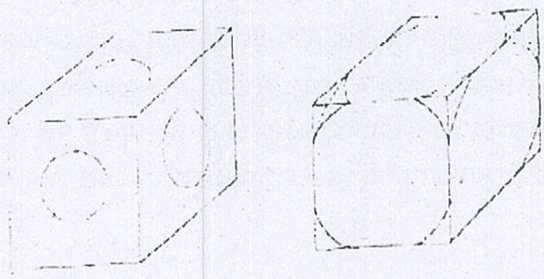

Figure 1 Boxed Beads: expansion stages

Under the action of the blowing agent the bead expands to a new volume $V_0 + dV$, where dV is prescribed. This expansion will give rise to 6 circular contact areas each of size dS where the sphere has fused to the side of the box. The bead surfaces not making contact with the box are assumed to be spherical of radius R say, a result born out by observations. Our aim is to determine dS(dV). For calculation purposes there are two stages to this process that need to be identified, see Figure 1.

Stage 1 (the early expansion stage):

The bubble makes contact with the sides of the box in circular regions not touching the edges of the box. For this stage the bead is spherical with n caps removed. From the geometry, see Figure 1, the volume of the constrained sphere is given by

$$V_0 + dV = \frac{4}{3}\pi R^3 - n(\frac{2}{3}\pi R^3 - \pi R^2 a_0 + \frac{1}{3}\pi a_0^3).$$

Scaling this equation with respect to V_0 and writing $r = R/a_0$, $v = dV/V_0$ we obtain

$$v = (r^3 - 1) - n(\frac{r^3}{3} - \frac{3r^2}{4} + \frac{1}{4}). \quad (2)$$

The total contact surface area will be

$$dS = n\pi(R^2 - a_0^2),$$

which after scaling ($s = dS/\pi a_0^2$) becomes

$$s = n(r^2 - 1). \quad (3)$$

Stage 2 (the later expansion stage):

For sufficiently large bubble volume (v>0.82 in the square box case) the bubble will flatten along the edges of the box, the contact surfaces will be circular patches with caps removed, and the bubble volume will be that of the box itself with corners lopped, see Figure 1. Evaluations are again straightforward but not simple and exact results appear to be unavailable. For the cube case after scaling (as above) we get (Equations (4) and (5)

$$1 + v = \frac{6}{\pi}(1 - I), \quad (4)$$

where

$$I = \int_0^{1-\sqrt{r^2-2}} g(r,y)dy, \text{ where}$$

$$g(r,y) = \int_0^{1-\sqrt{(r^2-1)-(y-1)^2}} [1-\sqrt{r^2-(y-1)^2-(x-1)^2}]dx,$$

and

$$s = (r^2-1)(1-\frac{4}{\pi}\arccos(\frac{1}{\sqrt{r^2-1}})) + \frac{4}{\pi}\sqrt{r^2-2}. \qquad (5)$$

Similar results have been obtained for the other Platonic polyhedra cases. When the volume of the bubble equals the volume of the box the contact surface area is the same as that of the box, and any further volumetric expansion of the bubble is impossible (v approaches $6/\pi$ 1 i.e. approximately 0.91 in the cube case, and s approaches $24/\pi$ i.e. approximately 7.64, again in the cube case).

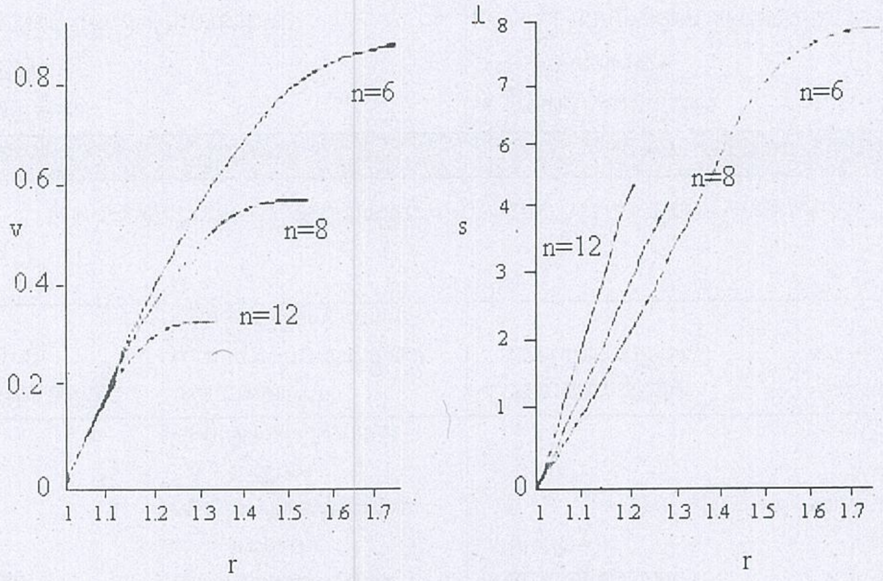

Figure 2. Bubble volume v(r) and contact surface area s(r) for boxes with n=6,8,12 sides.

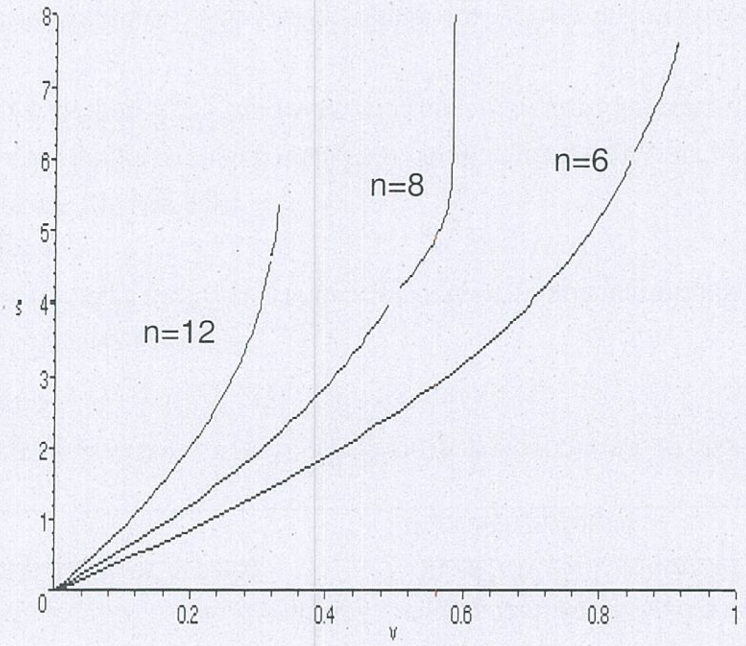

Figure 3. Surface area as a function of volume s(v) for boxes with n=6,8,12 sides

The s(r) and v(r) relationships are displayed in Figure 2, and the s(v) relationship in Figure 3 for all the polyhedra cases. Both expansion stages are covered. It should be noted that:

- As would be expected the contact surface area is much greater for dodecahedron packing than for octahedron packing; and this is in turn greater than that for the cube. Clearly this means that efficient packing can greatly influence product strength. A better packing might be realized in practice if the mould is vibrated during the moulding process.
- Once Stage 2 is reached there is a rapid increase in contact area with v. This suggests that, in order to ensure product strength, the total bead volume to mould volume ratio should be carefully set to ensure Stage 2 is reached during moulding.

Product Strength: Based on our strength model, the strength for a product consisting of just virgin beads is given by S_v= □ s(v), where □ is the bonding strength, determined experimentally. The equations (2,3,4,5) provide a parametric description for s(v) and thus the shearing or tensile strength of the moulded material formed under different manufacturing conditions can be determined or read off directly from the Figure 3. It should be recalled that v measures the relative volumetric expansion of the styrene beads *once the beads fill the rigid mould box.* Thus equivalently

$$v = \frac{V_M - V_B}{V_M}, \tag{6}$$

where V_B is the expanded volume of the beads used in the mould of volume V_M. The effect of the mould filling ratio on the strength of the material formed is thus determined.

2.3 Bead Mixtures

Virgin beads and reused beads initially contain different amounts of pentane, so that under the action of steam they expand by different amounts. Other properties are also different; for example the recycle beads are initially larger, having been previously expanded, and also have rougher and dimpled surfaces as a result of previous fusion followed by separation. The expansibility difference is, however, likely to be of dominant importance in context. Thus under the same mould pressure and temperature conditions we would not expect recycled beads to expand as much as the virgin beads. A simple way to model this situation is to assume that if the virgins expand by a volume v, then under the same conditions the recycled beads will expand by fv, where $1>f>0$ measures the quality of the recycled beads. The strength per unit area of contact appears to be the same for virgins and recycles, however the surface area of contact between virgin beads and recycled beads within the material produced under the moulding conditions will

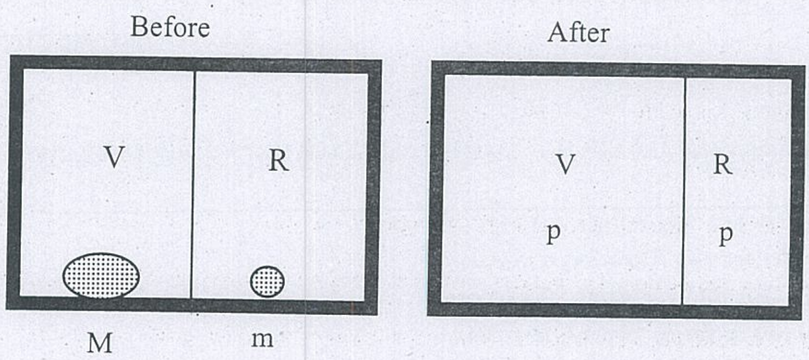

Figure 4: Expanding Beads

certainly be different to that for virgin virgin contacts and virgin recycled beads contact. The physics and geometry of the situation is best understood by examining what would happen in the simple situation depicted in Figure 4. We have two square cells of equal size within a solid rectangular mould. The cells contain different amounts (M,m) of pentane (simulating virgin and recycled cells) and are separated by an impermeable wall that moves in response to pressures generated within the two cells. The box is heated so that the pentane evaporates and the wall moves until the pressures in the two cells are the same. Evidently a greater proportion of the mould will be occupied by the virgin cell after expansion. The details of this situation can be worked out but are not important for our present purposes. Our bubbles are spherical and the surfaces of contact are the barriers, but the outcome will be essentially the same; equilibrium will be achieved when the pressures within virgins and reused beads are the same, with the virgins occupying more of the available space. Since the pressures are the same in both bead types the radius of free faces of the beads will be the same, however the contact areas will not normally be circular; it is a non-trivial exercise to determine the areas of contact. Calculations have been made for particular symmetric arrangements of virgin and recycled beads but will not be presented here. In the absence of an appropriate physical model the weighted average

$$s_{vr} = \frac{1+f}{2} s_{vv} \tag{7}$$

is the most obvious choice for the contact area between virgins and recycled beads.

Mixture Strength:

We are now in a position to determine the strength of the mixture as a function of f and the proportion of recycled beads. The contacts across any prescribed surface will be of three types; V-V, V-R and R-V, R-R, see Figure 5. If α is the proportion of recycled beads in the mixture then we expect α^2 of such links to be R-R in type, $2\alpha(1-\alpha)$ to be R-V in type, and the remaining $(1-\alpha)^2$ links to be V-V in type, so that

$$N_{vv} = (1-\alpha)^2 N_0, N_{rr} = \alpha^2 N_0, N_{vr} = 2\alpha(1-\alpha)N_0,$$

where N_0 is the total number of links across the surface.

All the terms of the strength equation(1) have now been modelled to give

$$S_m = s(v)[\alpha^2 + (1-\alpha)^2 f^2 + 2\alpha(1-\alpha)(1+f)/2].$$

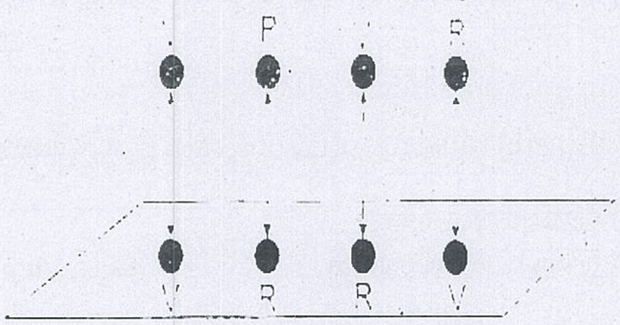

Figure 5: Links across a random slice

This provides the product strength vs `quality' , as eflected in f, and bead proportion, ⌉ result sought after. Note that as ⌉ tends to 1 S_m tends to S_v as expected. This result is plotted for various values of ⌉ and f in Figure 5. Evidently the strength is strongly influenced by both f and ⌉The appropriate s(v) relationship is determined by the effective number of contact faces n; the one best matches the product should be used. The calculations presented are for the n=6 cube case.

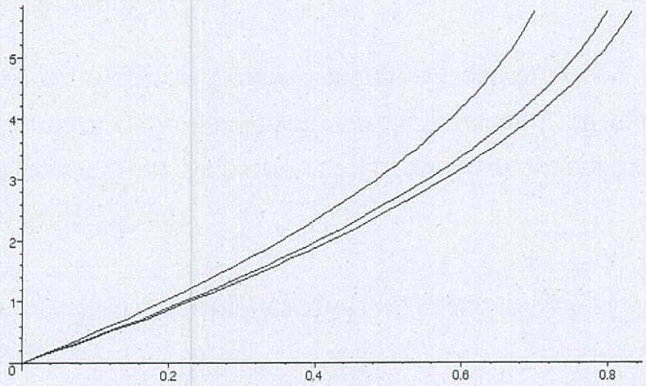

Figure 6: Strength for Mixtures s(v), with f=0.0and $\alpha = 0.6$ (lower curve), 0.8 and 1.0. n=6

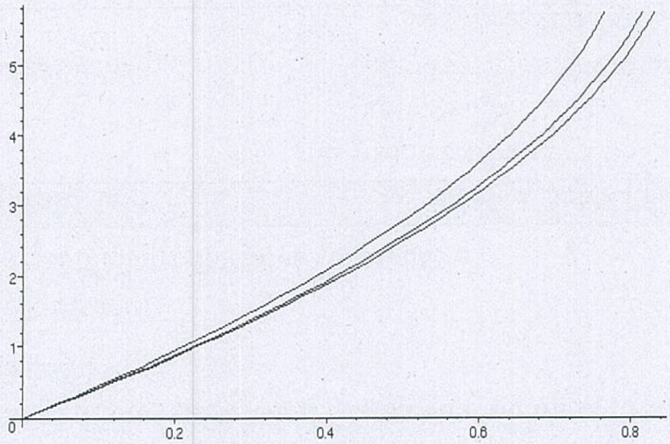

Figure 7. Strength for Mixtures s(v) with f=0.5, and ⌉ = 0.6, 0.8 and 1.0. n=6

2 Conclusions

The models above represent an attempt to understand and quantify the effect of composition on the structural strength of polystyrene slabs. The models are crude and in particular aspects not entirely adequate. The contact surface area calculations for bead mixtures are Particularly suspect. One suspects that an empirical approach is appropriate for determining this contact area, with the analytic results above being used as a framework.

3 Acknowledgements

This problem arose out of an Mathematics-in-Industry-Study-Group meeting held at QUT in February 1999 moderated by Frank deHoog and Warren Wood. Participants at this meeting all contributed to the understanding of the problem but the work of the moderators is especially acknowledged. The present article corrects and extends the work at this meeting.

4 References

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