KMITL Sci. J. Vol. 5 No. 1 Feb. 2005

# Solving Boundary Value Problem of Ordinary Differential Equations by Polynomials (2)

Maitree Podisuk and Pornchai Chaisanit
Department of Mathematics and Computer Science
King Mongkut's Institute of Technology Chaokhuntaharn Ladkrabang
Ladkrabang Bangkok 10520
THAILAND
mpodisuk@yahoo.com sinlp@yahoo.com http://www.kmitl.ac.th

Abstract: In this paper, we will use the polynomials of degree 12 to approximate the solution of boundary value problem of ordinary differential equations of order 4. We will use them to find the numerical solutions of some examples.

Key-Words: Polynomial Boundary-Value-Problem Ordinary Differential-Equation

## 1 Introduction

The boundary value problem of ordinary differential equation of second order may be written in the form

(1) $y''(x)+p(x)y'(x)+q(x)y(x)=r(x)$ for $x\in[a,b]$

with the boundary conditions .

(2) $y(a)=y_a$ and $y(b)=y_b$.

M. Podisuk and P. Chaisanit, [1], used the polynomial with the help of Guass Elimination Method to approximate the numerical solution of the above equations (1)-(2) instead of the Finite Difference Method and the results were very satisfactory. So in this paper we come up with the same idea but we will use this polynomial approximation method to approximate the numerical solution of the boundary value problem of the ordinary differential equations of order 4.

The boundary value problem of the ordinary differential equations of order 4 may be written in the form

(3) $y^{iv}(x)+p(x)y'''(x)+q(x)y''(x)+r(x)y'(x)+s(x)y(x)=f(x)$ for $x\in[a,b]$

with the boundary conditions

(4) $y(a)=c_1$, $y'(a)=c_2$, $y(b)=c_3$ and $y'(b)=c_4$.

If we divide the close interval $[a,b]$ into n subintervals at the points $x_0=a<x_1<x_2<...<x_n=b$, where $h=x_{i+1}-x_i$ for $i=0,1,2,...,n$ then we will come up with $n+3$ equations that is we may construct a polynomial of degree $n+2$ to approximate the numerical solution of the above equations (3)-(4).

## 2 Problem Formulation

At present time, the computer is every effective and very fast. Thus we should use this advantage to find the numerical solution of the boundary value problem of the ordinary differential equation. We shall call the method of our new idea as "Polynomial Approximation Method" which we shall denote by "PA". This idea is that we will use the polynomial

(5) $p_{n+2}(x)=a_0+a_1x+a_2x^2+...+a_{n+2}x^{n+2}$

to be the approximated polynomial for the unknown function y(x) of (1)-(2). We have

(6) $p'_{n+2}(x) = a_1 + 2a_2x + ... + (n+2)a_{n+2}x^{n+1}$

(7) $p''_{n+2}(x) = (2\cdot 1)a_2 + ... + (n+2)(n+1)a_{n+2}x^{n}$ .

(8) $p'''_{n+2}(x) = (3\cdot 2\cdot 1)a_3 + ... + (n+2)(n+1)na_{n+2}x^{n-1}$

(9) $p^{iv}_{n+2}(x) = (4\cdot 3\cdot 2\cdot 1)a_4 + ... + (n+2)(n+1)n(n-1)a_{n+2}x^{n-2}$

if we put (5),(6),(7),(8) and (9) in (3) then we obtain the equation

(10)
$$\begin{aligned}
&a_0\, s(x) + a_1(x\,s(x) + r(x)) + a_2(x^2\, s(x) + 2x\,r(x) + (2\cdot 1)\,q(x)) \\
&+ a_3(x^3\, s(x) + 3x^2\, r(x) + (3\cdot 2)x\,q(x) + (3\cdot 2\cdot 1)\,p(x) \\
&+ a_4(x^4\, s(x) + 4x^3\, r(x) + (4\cdot 3)x^2\, q(x) + (4\cdot 3\cdot 2)x^2\, p(x) + 4\cdot 3\cdot 2\cdot 1) + ... \\
&+ a_{n+2}(x^{n+2}\, s(x) + (n+2)x^{n+1}\, r(x) + (n+2)(n+1)x^{n}\, q(x) + (n+2)(n+1)n\,x^{n-1}\, p(x) \\
&+ (n+2)(n+1)n(n-1)) = f(x)
\end{aligned}$$

if we put $x_1, x_2, ..., x_{n-1}$ in (10) then we get $n-1$ equations. Now put $x_0 = a$ and $x_n = b$ in (5) and (6) we have the right hand side be $c_1, c_2, c_3$ and $c_4$. So we have four more equations. Thus we have the total of $n+3$ equations and $n+3$ unknowns. We use the Gauss Elimination method to solve this system and obtain the values $a_0, a_1, ..., a_{n+2}$ of polynomial $p_{n+2}(x)$ which we may use it to find the approximated value of unknown function $y(x)$ in the interval $[a,b]$.

## 3 Examples

There will be 3 examples in this section. We will use the above polynomial approximation method with $n = 10$ and the above finite difference method to find the numerical solutions of these 3 examples.

### 3.1 Example 1

Find the numerical solution of the equation

(11) $y^{iv} + y''' + y'' + y' + y = 5e^{x},\ x \in [0,2]$

with the initial condition

(12) $y(0) = 1$ , $y'(0) = 1$ , $y(2) = e^2$ and $y'(2) = e^2$ .

The analytical solution of the above equation is

$y(x) = e^{x}$ .

The numerical results of $y(x)$ , $y'(x)$ and $y''(x)$ by the polynomial approximation method are in the following table 1 , 2 and 3 for $h = 0.2$

| $x$ | Analytic solution of $y(x)$ | Approximate solution of $y(x)$ | Error |
|---|---|---|---|
| 0.2 | 1.2214027582 | 1.2214027582 | 0.0000000000 |
| 0.4 | 1.4918246976 | 1.4918246976 | 0.0000000000 |
| 0.6 | 1.8221188004 | 1.8221188004 | 0.0000000000 |
| 0.8 | 2.2255409285 | 2.2255409284 | 0.0000000001 |
| 1.0 | 2.7182818285 | 2.7182818284 | 0.0000000001 |
| 1.2 | 3.3201169227 | 3.3201169226 | 0.0000000001 |
| 1.4 | 4.0551999669 | 4.0551999668 | 0.0000000001 |
| 1.6 | 4.9530324244 | 4.9530324243 | 0.0000000001 |
| 1.8 | 6.0496474644 | 6.0496474644 | 0.0000000000 |

Table 1

| $x$ | Analytic solution of $y'(x)$ | Approximate solution of $y'(x)$ | Error |
|---|---|---|---|
| 0.2 | 1.2214027582 | 1.2214027581 | 0.0000000001 |
| 0.4 | 1.4918246976 | 1.4918246976 | 0.0000000000 |
| 0.6 | 1.8221188004 | 1.8221188003 | 0.0000000001 |
| 0.8 | 2.2255409285 | 2.2255409284 | 0.0000000001 |
| 1.0 | 2.7182818285 | 2.7182818284 | 0.0000000001 |
| 1.2 | 3.3201169227 | 3.3201169227 | 0.0000000000 |
| 1.4 | 4.0551999669 | 4.0551999669 | 0.0000000000 |
| 1.6 | 4.9530324244 | 4.9530324245 | 0.0000000001 |
| 1.8 | 6.0496474644 | 6.0496474646 | 0.0000000002 |

Table 2

| $x$ | Analytic solution of $y''(x)$ | Approximate solution of $y''(x)$ | Error |
|---|---|---|---|
| 0.2 | 1.2214027582 | 1.2214027577 | 0.0000000005 |
| 0.4 | 1.4918246976 | 1.4918246974 | 0.0000000002 |
| 0.6 | 1.8221188004 | 1.8221188003 | 0.0000000001 |
| 0.8 | 2.2255409285 | 2.2255409285 | 0.0000000000 |
| 1.0 | 2.7182818285 | 2.7182818286 | 0.0000000001 |
| 1.2 | 3.3201169227 | 3.3201169230 | 0.0000000003 |
| 1.4 | 4.0551999669 | 4.0551999672 | 0.0000000003 |
| 1.6 | 4.9530324244 | 4.9530324248 | 0.0000000004 |
| 1.8 | 6.0496474644 | 6.0496474649 | 0.0000000005 |

Table 3

**3.2 Example 2**

Find the numerical solution of the equation

(13) $$y^{iv}+\frac{1}{\pi^{3}}y'''+\frac{1}{\pi^{2}}y''+\frac{1}{\pi}y'+y=\pi^{4}\sin\pi x,\quad x\in[0,1]$$

with the initial condition

(14) $y(0)=0$ , $y'(0)=\pi$ , $y(1)=0$ and $y'(1)=-\pi$.

The analytical solution of the above equation is

$y(x)=\sin\pi x$.

The numerical results of $y(x)$ , $y'(x)$ and $y''(x)$ by the polynomial approximation method are in the following table 4 , 5 and 6 for $h=0.1$

| $x$ | Analytic solution of $y(x)$ | Approximate solution of $y(x)$ | Error |
|---|---|---|---|
| 0.1 | 0.3090169944 | 0.3090169747 | 0.0000000197 |
| 0.2 | 0.5877852523 | 0.5877851809 | 0.0000000714 |
| 0.3 | 0.8090169944 | 0.8090168511 | 0.0000001433 |
| 0.4 | 0.9510565163 | 0.9510562929 | 0.0000002234 |
| 0.5 | 1.0000000000 | 0.9999997000 | 0.0000003000 |
| 0.6 | 0.9510565163 | 0.9510561554 | 0.0000003609 |
| 0.7 | 0.8090169944 | 0.8090166000 | 0.0000003944 |
| 0.8 | 0.5877852523 | 0.5877848638 | 0.0000003885 |
| 0.9 | 0.3090169944 | 0.3090166630 | 0.0000003314 |

Table 4

| $x$ | Analytic solution of $y'(x)$ | Approximate solution of $y'(x)$ | Error |
|---|---|---|---|
| 0.1 | 2.9878321647 | 2.9878317881 | 0.0000003776 |
| 0.2 | 2.5416018462 | 2.5416012083 | 0.0000006379 |
| 0.3 | 1.8465818305 | 1.8465810503 | 0.0000007802 |
| 0.4 | 0.9708055194 | 0.9708047161 | 0.0000008033 |
| 0.5 | 0.0000000000 | -0.0000007073 | 0.0000007073 |
| 0.6 | -0.9708055194 | -0.9708060115 | 0.0000004921 |
| 0.7 | -1.8465818305 | -1.8465819883 | 0.0000001578 |
| 0.8 | -2.5416018462 | -2.5416015509 | 0.0000002953 |
| 0.9 | -2.9878321647 | -2.9878312980 | 0.0000008667 |

Table 5

| $x$ | Analytic solution of $y''(x)$ | Approximate solution of $y''(x)$ | Error |
|---|---|---|---|
| 0.1 | -3.0498754877 | -3.0498787012 | 0.0000032135 |
| 0.2 | -5.8012079129 | -5.8012099299 | 0.0000020170 |
| 0.3 | -7.9846776882 | -7.9846785156 | 0.0000008274 |
| 0.4 | -9.3865515789 | -9.3865512148 | 0.0000003641 |
| 0.5 | -9.8696044010 | -9.8696028452 | 0.0000015558 |
| 0.6 | -9.3865515789 | -9.3865488313 | 0.0000027476 |
| 0.7 | -7.9846776882 | -7.9846737506 | 0.0000039376 |
| 0.8 | -5.8012079129 | -5.8012027878 | 0.0000051251 |
| 0.9 | -3.0498754877 | -3.0498691890 | 0.0000062987 |

Table 6

### 3.3 Example 3

Find the numerical solution of the equation

(14) $$y^{iv}+\frac{3}{1+x}y'''+y''+\frac{1}{1+x}y'=0,\quad x\in[0,1]$$

with the initial condition

(15) $$y(0)=0\ ,\ y'(0)=1,\ y(1)=\ln 2 \text{ and } y'(1)=\frac{1}{2}.$$

The analytical solution of the above equation is
$y(x)=\ln(1+x)$.

The numerical results of $y(x)$, $y'(x)$ and $y''(x)$ by the polynomial approximation method are in the following table 7 , 8 and 9 for $h=0.1$

| $x$ | Analytic solution of $y(x)$ | Approximate solution of $y(x)$ | Error |
|---|---|---|---|
| 0.1 | 0.0953101798 | 0.0953101499 | 0.0000000299 |
| 0.2 | 0.1823215568 | 0.1823214550 | 0.0000001018 |
| 0.3 | 0.2623642645 | 0.2623640803 | 0.0000001842 |
| 0.4 | 0.3364722366 | 0.3364719842 | 0.0000002524 |
| 0.5 | 0.4054651081 | 0.4054648211 | 0.0000002870 |
| 0.6 | 0.4700036293 | 0.4700033566 | 0.0000002727 |
| 0.7 | 0.5306282511 | 0.5306280540 | 0.0000001971 |
| 0.8 | 0.5877866649 | 0.5877866142 | 0.0000000507 |
| 0.9 | 0.6418538862 | 0.6418540597 | 0.0000001735 |

Table 7

| $x$ | Analytic solution of $y'(x)$ | Approximate solution of $y'(x)$ | Error |
|---|---|---|---|
| 0.1 | 0.9090909091 | 0.9090903416 | 0.0000005675 |
| 0.2 | 0.8333333333 | 0.8333325162 | 0.0000008171 |
| 0.3 | 0.7692307692 | 0.7692299798 | 0.0000007894 |
| 0.4 | 0.7142857143 | 0.7142851712 | 0.0000005431 |
| 0.5 | 0.6666666667 | 0.6666665427 | 0.0000001240 |
| 0.6 | 0.6250000000 | 0.6250004322 | 0.0000004322 |
| 0.7 | 0.5882352941 | 0.5882363898 | 0.0000010957 |
| 0.8 | 0.5555555556 | 0.5555573982 | 0.0000018426 |
| 0.9 | 0.5263157895 | 0.5263184413 | 0.00000265181 |

Table 8

| $x$ | Analytic solution of $y''(x)$ | Approximate solution of $y''(x)$ | Error |
|---|---|---|---|
| 0.1 | -0.8264462810 | -0.8264504872 | 0.0000042062 |
| 0.2 | -0.6944444444 | -0.6944454305 | 0.0000009861 |
| 0.3 | -0.5917159763 | -0.5917145219 | 0.0000014544 |
| 0.4 | -0.5102040816 | -0.5102006874 | 0.0000033942 |
| 0.5 | -0.4444444444 | -0.4444395147 | 0.0000049297 |
| 0.6 | -0.3906250000 | -0.3906188578 | 0.0000061422 |
| 0.7 | -0.3460207613 | -0.3460136704 | 0.0000070909 |
| 0.8 | -0.3086419753 | -0.3086341699 | 0.0000078054 |
| 0.9 | -0.2770083103 | -0.2769999229 | 0.0000083874 |

Table 9

Note. Error in all tables is the absolute error compare with the analytical solution that is error $= |y_m - y(x_m)|$.

## 4 Conclusion

The polynomial approximation method works quite well. However in this paper we use the polynomial of degree 12. If we do not need very close approximation then we may reduce the degree of polynomial to be less than 12. In the other hand if we need more accuracy then we must use the polynomial of degree greater than 10 which will have no problem with most computer. The polynomial approximation method is not only work well but also has more advantage that we can approximate the value of the unknown function $y(x)$ at all points in the interval $[a,b]$ since we did not just obtain the approximated valued of the unknown function $y(x)$ at the points $x_1, x_2, \ldots, x_{n-1}$ but we obtained the polynomial $p_n(x)$ which we can use it to find the value of the unknown function $y(x)$ at all points in the interval $[a,b]$.

*Reference:*
[1] S.D. Conte and C. de Boor, Elementary Numerical Analysis second edition, McGraw-Hill Book Company, 1972.
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