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INTEGRATION METHOD FOR SOLVING THE INITIAL VALUE PROBLEM OF ORDINARY DIFFERENTIAL EQUATION

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ABSTRACT

In this paper, we will use numerical integration formula and Gauss elimination method to find numerical solution of the initial value problem of the ordinary differential equations of higher order by using the integral equation instead of the ordinary differential equation. We will use the trapezoidal formula to come up with the system of linear equations. Some examples will be illustrated.

KEYWORD: Gauss-elimination, Trapezoidal Linear-equations, Integral-equation

1. INTRODUCTION

The initial value problem of the ordinary differential equation of higher order is of the form

(1) $$F(x, y, y', \ldots, y^{(n)}) = 0$$

with the initial conditions

$$y(x_0) = c_1$$
$$y'(x_0) = c_2$$
(2) $$y''(x_0) = c_3$$
$$\vdots$$
$$y^{(n)}(x_0) = c_n .$$

The numerical method which most frequency use to find the numerical solution of the above system is the Runge-Kutta method by transforming the above system to be the system of n first order equations of initial value problem of ordinary differential equations as follows;

$$
(3) \quad
\begin{array}{ll}
y_1' = y_2, & y_1(x_0) = c_1 \\
y_2' = y_3, & y_2(x_0) = c_2 \\
y_3' = y_4, & y_3(x_0) = c_3 \\
\vdots & \\
y_{n-1}' = y_n, & y_{n-1}(x_0) = c_{n-1} \\
y_n' = F(x, y_1, y_2, y_3, \ldots, y_n), & y_n(x_0) = c_n.
\end{array}
$$

Then solve the above system of n first order equations for $y_1, y_2, y_3, \ldots, y_n$ simultaneously and the value of y_1 is the value of y in the original system.

2. PROBLEM FORMULATION

Nowadays, computer is so powerful that we can solve the large system in only few time. From this advantage, we then come up with the following idea of finding the numerical solution of the higher order of the initial value problem of ordinary differential equation.

We know that the first order of initial value problem of ordinary differential equation

(4) $$y'(x) = f(x, y), \qquad x \in [a, b]$$

with the initial condition

(5) $$y(a) = c$$

is equivalent to the integral equation

(6) $$y(x) = c + \int_a^x f(t, y)dt.$$

By the method of successive approximation, the sequence $\{y_n(x)\}$ converges to the solution of the above equations (4)-(5) where

(7) $$y_{m+1}(x) = c + \int_a^x f(t, y_m(t))dt.$$

So we will use the fact that the sequence $\{y_n(x)\}$ converges to the solution of the above equations (4)-(5) to approximate the numerical solution at the point $x = a + h$ by using the numerical approximation, trapezoidal formula,

(8) $$\int_a^{a+h} f(t, y(t))dt = \frac{h}{2}\left[f(a) + f(a+h)\right]$$

that is $y_{m+1} = y_m + \frac{h}{2}\left[y_m + y_{m+1}\right]$ or

(9) $$y_{m+1} = \frac{(1+0.5h)}{(1-0.5h)} y_m$$

We will use the above idea to apply to the fourth order initial value problem of ordinary differential equation

(10) $$y^{(4)} = p(x)y''' + q(x)y'' + r(x)y' + s(x)y + t(x), \qquad x \in [a, b]$$

with the initial conditions

(11) $y(a) = c_1, y'(a) = c_2, y''(a) = c_3$ and $y'''(a) = c_4$.

We obtain the 4 equations of first order of initial value problem of ordinary differential equation

(12) $$\begin{aligned} y_1' &= y_2, & y_1(a) &= c_1 \\ y_2' &= y_3, & y_2(a) &= c_2 \\ y_3' &= y_4, & y_3(a) &= c_3 \\ y_4' &= p(x)y_4 + q(x)y_3 + r(x)y_2 + s(x)y_1 + t(x), & y_4(a) &= c_4. \end{aligned}$$

Using equation (9), we obtain the system of linear equations

(13) $Ax = b$

where,

$$A = \begin{bmatrix} 1 & -\frac{h}{2} & 0 & 0 \\ 0 & 1 & -\frac{h}{2} & 0 \\ 0 & 0 & 1 & -\frac{h}{2} \\ -\frac{h}{2}s(x_{m+1}) & -\frac{h}{2}r(x_{m+1}) & -\frac{h}{2}q(x_{m+1}) & 1-\frac{h}{2}p(x_{m+1}) \end{bmatrix}$$

$$x = \begin{bmatrix} y_{1m+1} \\ y_{2m+1} \\ y_{3m+1} \\ y_{4m+1} \end{bmatrix} \text{ and}$$

$$b = \begin{bmatrix} y_{1m} + \frac{h}{2} y_{2m} \\ y_{2m} + \frac{h}{2} y_{3m} \\ y_{3m} + \frac{h}{2} y_{4m} \\ y_{4m} + \frac{h}{2} p(x_m) y_{4m} + \frac{h}{2} q(x_m) y_{3m} + \frac{h}{2} r(x_m) y_{2m} + \frac{h}{2} s(x_m) y_{1m} + \frac{h}{2} t(x_m) \end{bmatrix}$$

Then we may use the Gauss elimination method to find the values of $y_{1m+1}, y_{2m+1}, y_{3m+1}$ and y_{4m+1} which y_{1m+1} is the value of $y_{m+1} \approx y(x_{m+1})$. We shall call this method as **'Integration Method'**.

3. EXAMPLE

There will be three examples in this section.

Example 1 Find the numerical solution of the equations

(14) $y^{(4)} - 4y''' + 7y'' - 6y' + 2y = 2x^2 - 16x + 32 \text{ , } x \in [0,1]$

with the initial condition

$$y(0) = 3, y'(0) = -2, y''(0) = 1, y'''(0) = -2 .$$

The analytical solution of the above equation is

$$y(x) = e^x \cos x - e^x \sin x + xe^x - e^x + x^2 - 2x + 3 .$$

The numerical results are in the following table1, 2 and 3.

x=1.0 h=0.01			
x	Calculated y	y	error
0.1	2.8238029187	2.8238044729	$1.5541590983\times10^{-6}$
0.2	2.6357151797	2.6357185093	$3.3296309994\times10^{-6}$
0.3	2.4537005297	2.4537055251	$4.9953814596\times10^{-6}$
0.4	2.2756773164	2.2756836460	$6.3296683948\times10^{-6}$
0.5	2.0996476491	2.0996546935	$7.0444402809\times10^{-6}$
0.6	1.9237964939	1.9238032685	$6.7746805144\times10^{-6}$
0.7	1.7466142097	1.7466192770	$5.0673443184\times10^{-6}$
0.8	1.5670457722	1.5670471418	$1.3696117094\times10^{-6}$
0.9	1.3846701566	1.3846651735	$4.9830741773\times10^{-6}$
1.0	1.1999135684	1.1998987893	$1.4779127014\times10^{-5}$

Table 1

x=1.0 h=0.001			
x	Calculated y	y	error
0.1	2.8065733818	2.8065733989	$1.7142156139\times10^{-8}$
0.2	2.6191198348	2.6191198697	$3.4870626866\times10^{-8}$
0.3	2.4375506564	2.4375507077	$5.1295501180\times10^{-8}$
0.4	2.2597880028	2.2597880670	$6.4192136051\times10^{-8}$
0.5	2.0838457643	2.0838458350	$7.0631358540\times10^{-8}$
0.6	1.9079306325	1.9079306993	$6.6811480792\times10^{-8}$
0.7	1.7305668906	1.7305669388	$4.8166839406\times10^{-8}$
0.8	1.5507481951	1.5507482042	$9.0476532932\times10^{-9}$
0.9	1.3681198385	1.3681197812	$5.7269062381\times10^{-8}$
1.0	1.1831952019	1.1831950431	$1.5879959392\times10^{-7}$

Table 2

x=1.0 h=0.0001			
x	Calculated y	y	error
0.1	2.8046628157	2.8046628519	$1.3096723706 \times 10^{10}$
0.2	2.6172785459	2.6172785463	$3.8198777474 \times 10^{10}$
0.3	2.4359369427	2.4359369431	$3.7471181713 \times 10^{10}$
0.4	2.2581993357	2.2581993358	$9.8225427791 \times 10^{11}$
0.5	2.0822649693	2.0822649690	$2.9103830457 \times 10^{10}$
0.6	1.9063427949	1.9063427942	$6.2027538661 \times 10^{10}$
0.7	1.7289605960	1.728960594	$1.1514202924 \times 10^{-9}$
0.8	1.5491170269	1.5491170249	$1.9772414817 \times 10^{-9}$
0.9	1.3664641668	1.3664641637	$3.1632225728 \times 10^{-9}$
1.0	1.1815242974	1.1815242925	$4.8366928240 \times 10^{-9}$

Table 3

Example 2

Find the numerical solution of the equation

$$y^{(4)} - 2y'' + y = 8e^{x} + 4\sin x + 4\cos x \text{ , } x \in [0,1]$$

with the initial condition

$$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 5 .$$

The analytical solution of the above equation is

$$y(x) = x^2 e^x + \cos x + \sin x$$

The numerical results are in following table 4, 5 and 6.

x=1.0 h=0.01			
x	Calculated y	y	error
0.1	1.0946987881	1.0946940939	$4.6941986511 \times 10^{-6}$
0.2	1.2145295232	1.2145170406	1.2[illegible]85589 $\times 10^{-5}$
0.3	1.3566132871	1.3565896524	$2.3634742320 \times 10^{-5}$
0.4	1.5297852099	1.5297462537	$3.8956188291 \times 10^{-5}$
0.5	1.7449372774	1.7448778712	$5.9406176661 \times 10^{-5}$
0.6	2.0153561946	2.0152700694	$8.6125164671 \times 10^{-5}$
0.7	2.3571116294	2.3569911626	$1.2046674601 \times 10^{-4}$
0.8	2.7895023325	2.7893382977	$1.640370727 \times 10^{-4}$
0.9	3.335568690	3.3353499710	$2.1874595244 \times 10^{-4}$
1.0	4.0226815259	4.0223947457	$2.8678020317 \times 10^{-4}$

Table 4

x=1.0 h=0.001			
x	Calculated y	y	error
0.1	1.1058893445	1.1058892911	$5.3416442825 \times 10^{-8}$
0.2	1.2262744816	1.2262743482	$1.3337376004 \times 10^{-7}$
0.3	1.3707546113	1.3707543630	$2.4827568268 \times 10^{-7}$
0.4	1.5472100466	1.5472096410	$4.0567101678 \times 10^{-7}$
0.5	1.7667327975	1.7667321822	$6.1524769990 \times 10^{-7}$
0.6	2.0428421855	2.0428412969	$8.8863816927 \times 10^{-7}$
0.7	2.3918779423	2.3918767027	$1.2395830709 \times 10^{-6}$
0.8	2.8334519043	2.8334502200	$1.684333256 \times 10^{-6}$
0.9	3.3909669692	3.3909647271	$2.2421336325 \times 10^{-6}$
1.0	4.0922131964	4.0922102606	$2.9357761377 \times 10^{-6}$

Table 5

x=1.0 h=0.0001			
x	Calculated y	y	error
0.1	1.1058892916	1.1058892911	$5.4205884226 \times 10^{10}$
0.2	1.2275920203	1.2275920189	$1.3678800315 \times 10^{9}$
0.3	1.3721848851	1.3721848826	$2.4811015464 \times 10^{9}$
0.4	1.5489749342	1.5489749305	$3.7289282773 \times 10^{9}$
0.5	1.7689425471	1.7689425417	$5.4042175179 \times 10^{9}$
0.6	2.0456305826	2.0456305750	$7.5779098552 \times 10^{9}$
0.7	2.3954060994	2.3954060890	$1.0368239600 \times 10^{8}$
0.8	2.9379126154	2.8379126014	$1.3980752556 \times 10^{8}$
0.9	3.3965897114	2.3965896929	$1.8542777980 \times 10^{8}$
1.0	4.0992698753	4.0992698512	$2.4141627364 \times 10^{8}$

Table 6

Example 3

The equation

$$y_1'' = y_1 + 2y_2' - \frac{\lambda(y_1 + \mu)}{r_1^3} - \frac{(y_1 - \lambda)}{r_2^3}$$

$$y_2'' = y_2 - 2y_1' - \frac{\lambda y_1}{r_1^3} - \frac{\mu y_2}{r_2^3}$$

Defined by

$$r_1 = [(y_1 + \mu)^2 + y_2^2]^{\frac{1}{2}}$$

$$r_2 = [(y_1 - \lambda)^2 + y_2^2]^{\frac{1}{2}}$$

$$\mu \cong \frac{1}{82.45}$$

$$\lambda = 1 - \mu$$

with the initial condition

$$y_1(0)=1.2,\ y_1'(0)=0,\ y_2(0)=0,\ y_2'(0)=-0.9.$$

The numerical results with $h=0.005$ are in following figure 1.

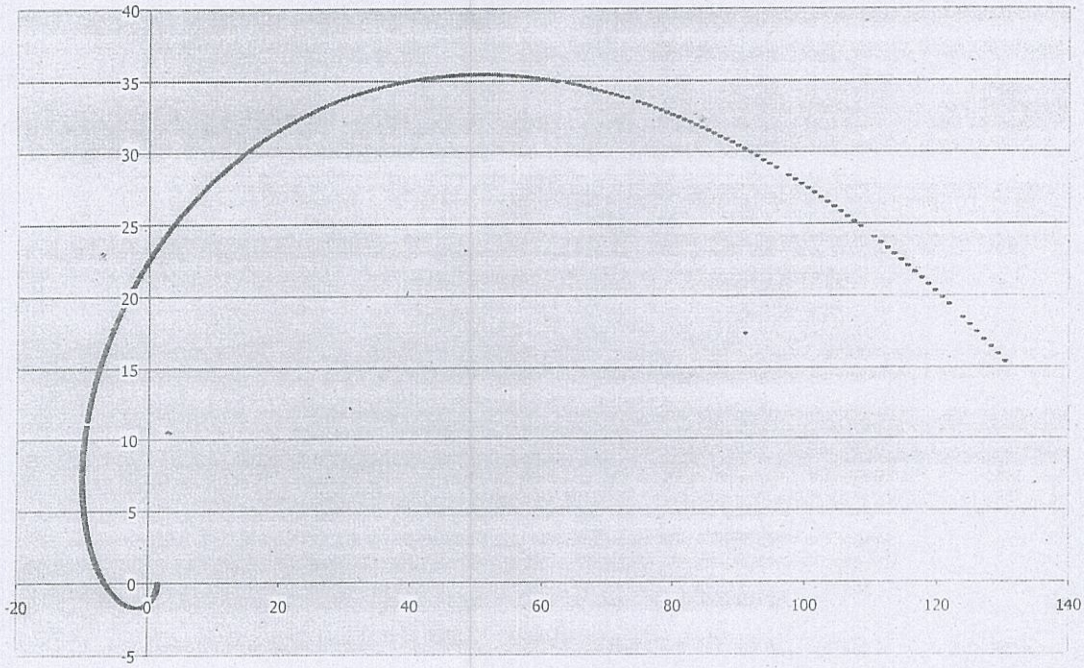

Figure 1

4. CONCLUSION

The new method, integration method, is the good numerical method for solving the initial value problem of ordinary differential equations. The results from the examples are satisfactory. This new integration method will give mathematician more freedom to select the way to find the numerical solution of the initial value problem of ordinary differential equations.

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