The Lyapunov Analyses of MERS-Cov Transmission in Thailand

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Abstract

This work investigates the transmission model of MERS-Cov using SEIR model which divides the total human population into four subclasses: susceptible, exposed, infected and recovered. Two equilibrium points were exhibited: the disease-free equilibrium E_1^* and the endemic equilibrium E_2^*. The basic reproduction number was computed via the next generation method. Two types of global stability of these equilibrium points were investigated through the theory of Lyapunov. Specifically, the exponential stability was investigated using a square type Lyapunov candidate function; while the asymptotic stability was investigated through a Logarithm type Lyapunov candidate function. It is theoretically shown that, when the reproductive number is less than unity. The disease-free equilibrium state is globally asymptotically stable, and the endemic equilibrium state is globally asymptotically stable if the reproductive number is greater than unity. Numerical results with parameters obtained from the previous work also illustrates the global asymptotical stability of the MERS-Cov system. These results can further be used for the design of a controller that drives the MERS-Cov system and the effective control reproductive number is less than 1 so that the stability of the controlled system would be similar to that of the uncontrolled disease-free system.

Keywords: Global dynamical modeling method, Lyapunov function method, MERS, transmission model, basic reproductive number, next generation method, disease-free equilibrium state, endemic equilibrium state
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1. Introduction

MERS-Cov is the short name for Middle East Respiratory Syndrome. Cov has been added since the disease was caused by a corona virus. It was first detected in 2012 in Saudi Arabia as a case of severe viral pneumonia which developed to a case of acute respiratory distress syndrome and death [1]. In Thailand, the first case was a 75 years old male from Oman, who was reported with the symptoms on 18 June 2015. On 23 January 2016, a second case from an Arab person was reported.The third case was reported in a person from Kuwait on 30 July 2016.MERS-Cov transmission has been found in many countries around the world; Asia, Europe, the United States and North Africa. World Health Organization (WHO) reported 2,123 laboratory-confirmed infection cases with corona virus world wide, including at least 740 deaths in 27 countries during September 2012 to January 2018 [2].

On 23 August 2017, the one additional Middle East Respiratory Syndrome (MERS-Cov) case was reported by the national IHR focal point of the United Arab Emirates (UAE) [3].The risk assessment by WHO indicates that MERS-Cov causes severe human infections resulting in high mortality and has demonstrated the ability to transmit between humans. So far, the observed human-to-human transmissions have occurred mainly in health care settings. Figure 1 shows the confirmed cases of infections and deaths in 2016, with peaks around March, July and October. It is implied that MERS-Cov infection cases occur worldwide.

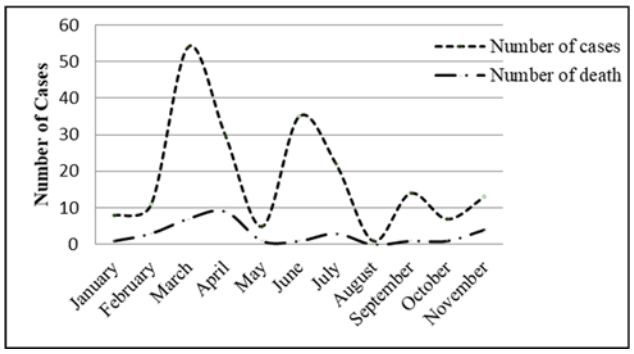


Figure 1. Confirmed global cases and death of MERS from 1 January 2016 to15 November 2016 [4].

In 2013, Aburizaiza *et al.*[5] investigated the Anti–Middle East Respiratory Syndrome Antibodies in Blood Donors and Slaughterhouse workers in Jeddah and Makkah. Their investigation looked for the antibodies using immune fluorescence assay (IFA) , a differential recombinant IFA, and a plaque-reduction serum neutralization assay. Altogether, 130 blood donors were sampled during 2012 in Jeddah and 226 slaughter house workers were sampled in October 2012 in Jeddah and Makkah, Saudi Arabia. In 2014, Chowell *et al.* [6] formulated a mathematical model to study the outbreak of MERS-Cov during April-October 2013 in Arabian Peninsula. Kim *et al.* [7] studied MERS-Cov transmission dynamics in South Korea, resulting in the formulation of SEIAHR model (susceptible-exposed- infectious- infection but asymptomatic-hospitalized-recovery). They found the basic reproductive number in two periods. First period was very large due to the superspreader. The second period was reduced significantly after intensive interventions to reduce the basic reproduction number. The literature reviews arouses our interests in the development of the mathematical model and Lyapunov stability analyses. This work considers the global dynamical transmission model, where three types of stability, namely, local, exponential and asymptotic stabilities were investigated through the use of the Lyapunov function

candidates. Specifically, the square-type Lyapunov functions were used to show the exponential stability of the equilibrium points, while the logarithmic-type Lyapunov candidates was chosen to illustrate the global asymptotical stability of the equilibrium points. The next generation method was also applied to compute the basic reproductive number.

2. Formulation of the Model and Stability Analyses

In this model, we study Susceptible-Exposed-Infected-Recovered (or SEIR) model to describe the dynamical transmission of MERS in Thailand.We assume that the transmission of corona virus is possible only through the pathway of person-to-person contacts, not through the camel-to-person contacts, which would be possible if we were considering the situation in Saudi Arabia. The human population is classified into four sub-classes: susceptible individuals (S^h), exposed individuals (E^h), infected individuals (I^h) and recovered individuals (R^h). The total population of the human at time t is denoted $N^h(t)$ where $N^h(t) = S^h(t) + E^h(t) + I^h(t) + R^h(t)$. Human recruitment rate is denoted as B and δdenotes the rate at human-to-human MERS-Cov contract occurs. The rate of expose for susceptible human is given by $\frac{\delta S^h I^h}{N^h}$. The rate at which the exposed human become infected is denoted as φ. Recovery rate of human with MERS-Cov is γ. where μ and α are the natural death rate and the additional disease death rate due to the MERS-Cov infection, respectively. The number of members in the susceptible class is increased by the human recruitment rate B and reduced by infection and the natural death. The exposed population is increased by the infection of a susceptible, but is reduced through natural death. The infected population increases when an exposed person becomes infectious but is diminished by recovery from the disease, natural death and additional disease death. The recovered population is increased by the recovery of infected person and decreased by a natural death. The transmission schematic is shown in Figure 2.

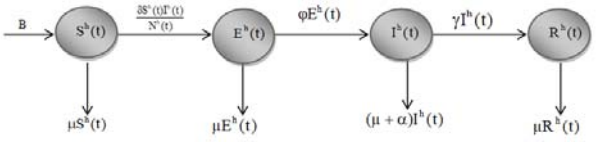


Figure 2. Flow chart of transmission model of human population

The transmission flow chart admits the following system of differential equations defined:

$$\frac{dS^h}{dt} = B - \left(\mu + \frac{\delta I^h}{N^h}\right) S^h \tag{1}$$

$$\frac{dE^h}{dt} = \frac{\delta S^h I^h}{N^h} - (\mu + \varphi) E^h \tag{2}$$

$$\frac{dI^h}{dt} = \varphi E^h - (\mu + \alpha + \gamma) I^h \tag{3}$$

$$\frac{dR^h}{dt} = \gamma I^h - \mu R^h \tag{4}$$

The non-negativeness of the solutions

Proposition 1 Let $(S^h(t), E^h(t), I^h(t), R^h(t))$ be the solutions of equations (1)-(4).

Denoting also the invariant set $\phi = \left\{(S^h, E^h, I^h, R^h) \in R_+^4 : N^h \le \frac{B}{\mu}\right\}$. Then the closed set ϕ is positive invariant.

Proof We begin by setting $N^h(t) = (S^h + E^h + I^h + R^h)$ and assume that $N^h = \frac{B}{\mu}$. Note that the total population N^h is non-negative definite on R_+^4 Then we have:

$$\frac{dN^h}{dt} = \frac{dS^h}{dt} + \frac{dE^h}{dt} + \frac{dI^h}{dt} + \frac{dR^h}{dt}$$

$$\frac{dN^h}{dt} = B - \left(\mu + \frac{\delta I^h}{N^h}\right) S^h + \frac{\delta S^h I^h}{N^h} - (\mu + \varphi) E^h + \varphi E^h - (\mu + \alpha + \gamma) I^h + \gamma I^h - \mu R^h$$

$$\frac{dN^h}{dt} = B - \mu N^h - \alpha I^h \le B - \mu N^h$$

Then it follows that $\frac{dN^h}{dt} \le 0$ on $0 \le N^h(t) \le N^h(0) e^{-\mu t} + \frac{B}{\mu}(1 - e^{-\mu t})$. As $t \to \infty, e^{-\mu t} \to \infty$ and we have $\lim_{t \to \infty} N^h(t) \le \frac{B}{\mu}$, $N^h(t)$ approaches $\frac{B}{\mu}$. Since the region of all solutions of ϕ is in R_+^4.

2.1 Equilibrium Points

Proposition 2 Equations (1)-(4) have two equilibrium points: for $R_0 \le 1$ the equilibrium points is the disease free steady state $E_1^*(S^{h*}, E^{h*}, I^{h*}, R^{h*}) = \left(\frac{B}{\mu}, 0, 0, 0\right) \in \phi$. For $R_0 > 1$, the equilibrium point is the endemic steady state $E_2^*(S^{h*}, E^{h*}, I^{h*}, R^{h*}) \in \phi$ and satisfies $S^{h*}, E^{h*}, I^{h*}, R^{h*} > 0$, where

$$S^{h*} = \frac{BN^h}{\mu N^h + \delta I^{h*}} \tag{5}$$

$$E^{h*} = \frac{B\delta I^{h*}}{(\mu + \psi)(\delta I^{h*} + \mu N^h)} \tag{6}$$

$$I^{h*} = \frac{B\delta\psi - \mu N^h(\alpha + \gamma + \mu)(\mu + \psi)}{\delta(\alpha + \gamma + \mu)(\mu + \psi)} \tag{7}$$

and

$$R^{h*} = \frac{\gamma I^{h*}}{\mu} \tag{8}$$

Proof Steady states $(S^{h*}, E^{h*}, I^{h*}, R^{h*})$ of our equations are found by setting equations (1)-(4) to zero.

$$B - \left(\mu + \frac{\delta I^h}{N^h}\right) S^h = 0 \tag{9}$$

$$\frac{\delta S^h I^h}{N^h} - (\mu + \varphi)E^h = 0 \tag{10}$$

$$\varphi E^h - (\mu + \alpha + \gamma)I^h = 0 \tag{11}$$

$$\gamma I^h - \mu R^h = 0 \tag{12}$$

From equation (9) we have $S^{h^*} = \frac{BN^h}{\mu N^h + \delta I^{h^*}}$

Equation (10) implies:

$$E^{h^*} = \frac{\delta S^{h^*} I^{h^*}}{N^h(\mu + \varphi)}$$

$$E^{h^*} = \frac{\delta\left(\frac{BN^h}{\mu N^h + \delta I^{h^*}}\right) I^{h^*}}{N^h(\mu + \varphi)}$$

$$E^{h^*} = \frac{B\delta I^{h^*}}{(\mu + \varphi)(\delta I^{h^*} + \mu N^h)}$$

Equation (11) implies:

$$I^{h^*} = \frac{\varphi E^{h^*}}{\alpha + \gamma + \mu}$$

$$I^{h^*} = \frac{\varphi\left(\frac{B\delta I^{h^*}}{(\mu + \varphi)(\delta I^{h^*} + \mu N^h)}\right)}{\alpha + \gamma + \mu}$$

$$I^{h^*} = \frac{B\delta\varphi - \mu N^h(\alpha + \gamma + \mu)(\mu + \varphi)}{\delta(\alpha + \gamma + \mu)(\mu + \varphi)}$$

Equation (12) implies:

$$R^{h^*} = \frac{\gamma I^{h^*}}{\mu}$$

Remark 1 The local stability of E_1^* is obtained by next generation matrix [8-9]. We identify classes E and I as being relevant. The disease-free steady state $E_1^*(S^{h^*}, E^{h^*}, I^{h^*}, R^{h^*}) = \left(\frac{B}{\mu}, 0, 0, 0\right)$

Gains and losses:

$$\begin{vmatrix} Gains\,to\,E: & (\delta S^h I^h)/N^h \\ Gains\,to\,I: & 0 \\ Losses\ from\,E: & (\mu + \varphi)E^h \\ Losses\ from\,I: & -\varphi E^h + (\mu + \alpha + \varphi)I^h \end{vmatrix}$$

The F (gains) and V (losses) matrices

$$F = \begin{bmatrix} \frac{\partial}{\partial E^h}\left(\frac{\delta S^h I^h}{N^h}\right) & \frac{\partial}{\partial E^h}(0) \\ \frac{\partial}{\partial I^h}\left(\frac{\delta S^h I^h}{N^h}\right) & \frac{\partial}{\partial I^h}(0) \end{bmatrix}$$

$$V=\begin{bmatrix} \frac{\partial}{\partial E^h}\left((\mu+\varphi)E^h\right) & \frac{\partial}{\partial E^h}(-\varphi E^h+(\mu+\alpha+\varphi)I^h) \\ \frac{\partial}{\partial I^h}\left((\mu+\varphi)E^h\right) & \frac{\partial}{\partial I^h}(-\varphi E^h+(\mu+\alpha+\varphi)I^h) \end{bmatrix}$$

Then we have

$$F=\begin{bmatrix} 0 & 0 \\ \frac{\delta S^h}{N^h} & 0 \end{bmatrix} \Rightarrow F=\begin{bmatrix} 0 & 0 \\ \frac{B\delta}{\mu N^h} & 0 \end{bmatrix}$$

$$V=\begin{bmatrix} \mu+\varphi & -\varphi \\ 0 & \mu+\alpha+\varphi \end{bmatrix}$$

$$V^{-1}=\frac{1}{(\mu+\varphi)(\mu+\alpha+\varphi)}\begin{bmatrix} \mu+\alpha+\varphi & \varphi \\ 0 & \mu+\varphi \end{bmatrix}$$

$$V^{-1}=\begin{bmatrix} \frac{1}{\mu+\varphi} & \frac{\varphi}{(\mu+\varphi)(\mu+\alpha+\varphi)} \\ 0 & \frac{1}{\mu+\alpha+\varphi} \end{bmatrix}$$

$$G=FV^{-1}$$

$$G=\begin{bmatrix} 0 & 0 \\ \frac{B\delta}{\mu N^h} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{\mu+\varphi} & \frac{\varphi}{(\mu+\varphi)(\mu+\alpha+\varphi)} \\ 0 & \frac{1}{\mu+\alpha+\varphi} \end{bmatrix}$$

$$G=\begin{bmatrix} 0 & 0 \\ \frac{B\delta}{\mu N^h(\mu+\varphi)} & \frac{\varphi B\delta}{\mu N^h(\mu+\varphi)(\mu+\alpha+\varphi)} \end{bmatrix}$$

R_0 is the dominant eigenvalue of the matrix $G=FV^{-1}$. Then we have

$$R_0=\frac{\varphi B\delta}{\mu N^h(\mu+\varphi)(\mu+\alpha+\varphi)} \tag{13}$$

2.2 Exponential stabilities of the equilibrium states

Theorem 1 Let $E_1^*=(S^{h^*},E^{h^*},I^{h^*},R^{h^*})=\left(\frac{B}{\mu},0,0,0\right)$, and $\alpha_3=\max\left(\mu+\frac{\delta I}{N},\mu+\gamma\right)$. The disease free equilibrium E_1^* is exponentially stable in the $\frac{dF}{dt}=B-\mu N^h-\alpha I^h\le B-\mu N^h$ when $R_0<1$.

Proof Consider the following Lyapunov candidate:

$$V_1=\frac{1}{2}\left(S^h\right)^2+\frac{1}{2}\left(E^h\right)^2+\frac{1}{2}\left(I^h\right)^2+\frac{1}{2}\left(R^h\right)^2 \tag{14}$$

Let $\mathbf{x}\equiv\begin{bmatrix} S^h & E^h & I^h & R^h \end{bmatrix}^T$ be the vector of states. It is obvious from the Lyapunov candidate V_1 that the following in equality is satisfied:

$$\alpha_1\|\mathbf{x}\|^2\le V_1(\mathbf{x})\le\alpha_2\|\mathbf{x}\|^2 \tag{15}$$

where α_1 and α_2 are both unity. Consider differentiating the Lyapunov candidate in the trajectory of the system:

$$V_1'(\mathbf{x}) = -\beta S^h - \left(\mu - \frac{\delta I}{N}\right)\left(S^h\right)^2 - \frac{E^h \delta S^h I^h}{N} - \left(\mu+\gamma\right)\left(E^h\right)^2 + I^h E^h \varphi - \left(\mu+\alpha+\gamma\right)\left(I^h\right)^2 + \gamma IR - \mu\left(R^h\right)^2$$
$$\leq \alpha_2\left(\left\|S^h\right\|^2 + \left\|E^h\right\|^2 + \left\|I^h\right\|^2 + \left\|R^h\right\|^2\right)$$
$$\leq \alpha_3\left\|\mathbf{x}\right\|^2 \tag{16}$$

Hence it follows that the differential inequality:

$$V'(\mathbf{x}) \leq \frac{\alpha_3}{\alpha_2} V(\mathbf{x}) = \alpha_3 V(\mathbf{x}) \tag{17}$$

or $V(\mathbf{x}) \leq |V(\mathbf{x}_0)| e^{-\alpha_3 t}$ which hereby implies the exponential stability of the MERS-Cov system.

Theorem 2 Let $E_2^* = (S^{h*}, E^{h*}, I^{h*}, R^{h*})$, and that $\gamma = \frac{\delta S^{h*}}{N}$, $\mu = \frac{B}{S^{h*}}$. The endemic equilibrium E_2^* is exponentially stable in the $\frac{dF}{dt} = B - \mu N^h - \alpha I^h \leq B - \mu N^h$ when $R_0 > 1$ with $\beta_2 = \max\left(\left(\mu + \frac{\delta I}{N}\right), \frac{\varphi\beta\delta}{\mu N(\mu+\varphi)R_0}\right)$.

Proof Consider the following Lyapunov candidate:

$$V_2 = \frac{1}{2}(S - S^*)^2 + \frac{1}{2}E^2 + \frac{1}{2}I^2 \tag{18}$$

The derivative of $V_2(\mathbf{x})$ in the trajectory of the model is given by:

$$V_2' = BS^h - \left(\mu + \frac{\delta I^h}{N^h}\right)\left(S^h\right)^2 - S^*\left(B - \left(\mu + \frac{\delta I^h}{N^h}\right)S^h\right) - \frac{E^h \delta I^h}{N^h} - \left(\mu+\gamma\right)\left(E^h\right)^2 + \varphi I^h E^h - \left(\mu+\alpha+\gamma\right)\left(I^h\right)^2$$
$$\leq -\left(\mu + \frac{\delta I^h}{N^h}\right)\left(S^h\right)^2 - \left(\mu+\gamma\right)\left(E^h\right)^2 - \left(\mu+\alpha+\gamma\right)\left(I^h\right)^2 - \mu\left(R^h\right)^2$$
$$\leq \beta_2\left(\left(S^h\right)^2 + \left(E^h\right)^2 + \left(I^h\right)^2 + \left(R^h\right)^2\right) \tag{19}$$

where $\beta_2 = \max\left(\left(\mu + \frac{\delta I}{N}\right), \frac{\varphi\beta\delta}{\mu N(\mu+\varphi)R_0}\right)$. It then follows again that $V_2'(\mathbf{x}) \leq \frac{\beta_3}{\beta_2} V_2(\mathbf{x}) = \beta_3 V_2(\mathbf{x})$, or $V_2(\mathbf{x}) \leq |V_2(\mathbf{x}_0)| e^{-\beta_3 t}$ which thus implies exponential stability of the system.

2.3 Global stability of the equilibrium states

Theorem 3 Let $E_1^* = (S^{h*}, E^{h*}, I^{h*}, R^{h*}) = \left(\frac{B}{\mu}, 0, 0, 0\right)$, and $\alpha = \frac{B\delta}{\mu N^h}$. The disease free equilibrium E_1^* is globally asymptotically stable in the $\frac{dF}{dt} = B - \mu N^h - \alpha I^h \leq B - \mu N^h$ when $R_0 < 1$.

Proof We consider a Lyapunov function $P(S^h, E^h, I^h, R^h) = (S^h - S^{h*}\ln S^h) + E^h + I^h + R^h$, then we have $P(S^{h*}, E^{h*}, I^{h*}, R^{h*}) = P\left(\frac{B}{\mu}, 0, 0, 0\right) = 0$ and the derivative with respect to time in the trajectory of the system yields:

$$\frac{dP}{dt} = S^{h\prime}\left(1 - \frac{S^{h*}}{S^h}\right) + E^{h\prime} + I^{h\prime} + R^{h\prime}$$

$$\frac{dP}{dt}=\left(B-\mu S^h-\frac{\delta S^h I^h}{N^h}\right)\left(1-\frac{S^{h*}}{S^h}\right)+\frac{\delta S^h I^h}{N^h}-(\mu+\varphi)E^h+\varphi E^h-(\mu+\alpha+\gamma)I^h+\gamma I^h-\mu R^h$$

$$\frac{dP}{dt}=\left(B-\mu S^h-\frac{\delta S^h I^h}{N^h}\right)\left(1-\frac{S^{h*}}{S^h}\right)+\frac{\delta S^h I^h}{N^h}-\mu E^h-\varphi E^h+\varphi E^h-\mu I^h-\alpha I^h-\gamma I^h+\gamma I^h-\mu R^h$$

$$\frac{dP}{dt}=\left(B-\mu S^h-\frac{\delta S^h I^h}{N^h}\right)\left(1-\frac{S^{h*}}{S^h}\right)+\frac{\delta S^h I^h}{N^h}-\alpha I^h-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=B\left(1-\frac{S^{h*}}{S^h}\right)-\mu S^h+\mu S^{h*}+\frac{\delta I^h S^{h*}}{N^h}-\alpha I^h-\mu E^h-\mu I^h-\mu R^h$$

Substituting $S^{h*}=\frac{B}{\mu}$ yields:

$$\frac{dP}{dt}=B\left(1-\frac{S^{h*}}{S^h}\right)-\mu S^h+\frac{\mu B}{\mu}+\frac{B\delta I^h}{\mu N^h}-\alpha I^h-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=B\left(1-\frac{S^{h*}}{S^h}\right)-\mu S^h+B-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=B\left(1-\frac{S^{h*}}{S^h}\right)-\frac{B\mu S^h}{B}+B-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=B\left(1-\frac{S^{h*}}{S^h}\right)+B\left(1-\frac{\mu S^h}{B}\right)-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=B\left(1-\frac{S^{h*}}{S^h}\right)+B\left(1-\frac{S^h}{S^{h*}}\right)-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=B\left(2-\frac{S^{h*}}{S^h}-\frac{S^h}{S^{h*}}\right)-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=B\left(\frac{2S^{h*}S^h-S^{h*2}-S^{h2}}{S^{h*}S^h}\right)-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=-B\left(\frac{(S^{h*}-S^h)^2}{S^{h*}S^h}\right)-\mu E^h-\mu I^h-\mu R^h$$

$$\frac{dP}{dt}=-\left[B\left(\frac{(S^{h*}-S^h)^2}{S^{h*}S^h}\right)+\mu E^h+\mu I^h+\mu R^h\right]<0 \qquad (20)$$

Then we have $\frac{dP}{dt}\le 0$ and all terms in Equation (20) are non-positive with $\frac{dP}{dt}=0$ if and only if $S^{h*}=S^h, E^h=0, I^h=0$ and $R^h=0$ in the equations (1)-(4). Using LaSalle' s extension to Lyapunov's method [10], the solution does not exist for $E_1^*(S^{h*},E^{h*},I^{h*},R^{h*})\in\phi$. Therefore, the globally asymptotically stable of the disease-free equilibrium E_1^* is satisfied.

Theorem 4 Let $E_2^*=(S^{h*},E^{h*},I^{h*},R^{h*})$. If $R_0>1$, then the endemic equilibrium point E_2^* is globally stable in the $\frac{dF}{dt}=B-\mu N^h-\alpha I^h\le B-\mu N^h$

Assume that

$$\begin{cases} \gamma = \dfrac{\delta S^{h^*}}{N^h} \\ \mu = \dfrac{B}{S^{h^*}} \end{cases}$$

Proof Let a Lyapunov function candidate be $K(S^h, E^h, I^h, R^h) = (S^h - S^{h^*} \ln S^h) + E^h + I^h$.

Then we have

$$\frac{dK}{dt} = (S^{h\prime} - \frac{S^{h^*}}{S^h} S^{h\prime}) + E^{h\prime} + I^{h\prime}$$

$$\frac{dK}{dt} = \left(B - \mu S^h - \frac{\delta S^h I^h}{N^h} \right) \left(1 - \frac{S^{h^*}}{S^h} \right) + \frac{\delta S^h I^h}{N^h} - (\mu + \varphi) E^h + \varphi E^h - (\mu + \alpha + \gamma) I^h$$

$$\frac{dK}{dt} = B\left(1 - \frac{S^{h^*}}{S^h} \right) - \mu S^h + \mu S^{h^*} - \mu E^h - \mu I^h - \alpha I^h$$

$$\frac{dK}{dt} = B\left(1 - \frac{S^{h^*}}{S^h} \right) - \frac{BS^h}{S^{h^*}} + \frac{BS^{h^*}}{S^{h^*}} - \mu E^h - \mu I^h - \alpha I^h$$

$$\frac{dK}{dt} = B\left(1 - \frac{S^{h^*}}{S^h} \right) + B\left(1 - \frac{S^h}{S^{h^*}} \right) - \mu E^h - \mu I^h - \alpha I^h$$

$$\frac{dK}{dt} = -B\frac{(S^{h^*} - S^h)^2}{S^h S^{h^*}} - \mu E^h - \mu I^h - \alpha I^h$$

$$\frac{dK}{dt} = -\left[B\frac{(S^{h^*} - S^h)^2}{S^h S^{h^*}} + \mu E^h + \mu I^h + \alpha I^h \right] < 0, \forall (S^{h^*}, E^{h^*}, I^{h^*}, R^{h^*}) \in \phi \tag{21}$$

Hence, the derivative $\frac{dK}{dt} \le 0, \forall (S^{h^*}, E^{h^*}, I^{h^*}, R^{h^*}) \in \phi$ with $\frac{dK}{dt} = 0$ if and only if $S^{h^*} = S^h, E^h = 0$ and $I^h = 0$ in the equations (1)-(4). Hence by LaSalle's extension to Lyapunov's method [10], the endemic equilibrium state E_2^* is globally asymptotically point on ϕ.

3. Results and Discussion

In this section, we give some numerical results of the presented system. The parameters are taken from the work of Lamwong *et al.* [11] for the Thai population only.

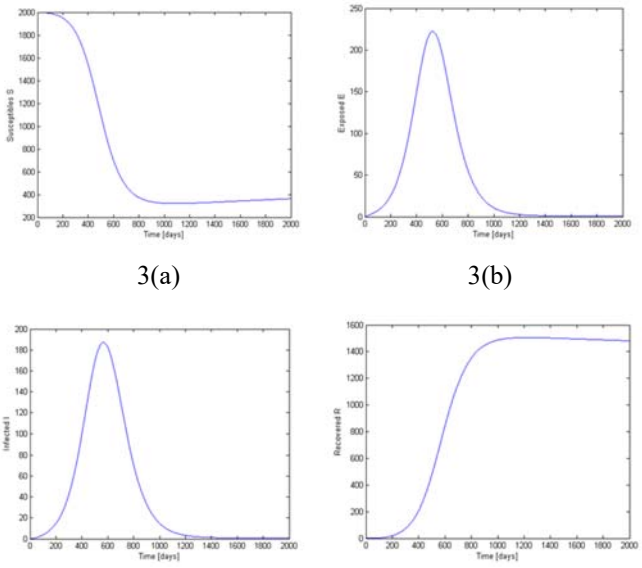
Table 1. Parameters used in the numerical simulation [11]

Parameter name	Value
B	$1/(75 \times 365)$
N^h	2000
δ	0.05
μ	$1/(75 \times 365)$
φ	1/50
α	1/365
γ	1/50

The numerical solutions with the initial condition of $\left[N^h - 2, 0, 0, 0 \right]^T$ are plotted in the graphs as shown in Figure 3(a-d). Form Figure 3(a). the trajectory of the susceptibles starts at a value close to 2000, which then decays and steadies at around 400 around t=1000 days. The plots of the exposed and infected populations start at zeros, peaking at around t=600 days, before reaching the steady state at a value near zero at around t =1000 days. The plot of the recovered population response in Figure 3(d) shows the reverse case of the plot of Figure 3(a). Specifically, there is zero initial response in the recovered population compartment. This number then increases exponentially and is steady at approximately 1400 after 1000 days. The steady state being reached by each compartments of the system is a nonzero number, thereby depicting an endemic equilibrium which is asymptotically stable.

4. Conclusions

In this work, we analyzed the standard dynamical modeling method where both types of typical Lyapunov candidate functions used for investigating epidemiological models were chosen. An exponential stability was investigated with the use of the square-type Lyapunov candidate, while the logarithm type Lyapunov candidate was chosen to show the global asymptotic stability of the equilibrium points. The model exhibits two equilibrium point, namely the disease-free steady state E_1^* and the endemic steady state E_2^*. The basic reproductive number is calculated by using the next generation method. If basic reproductive number is less than one, the disease-free equilibrium state is globally asymptotically stable. The endemic equilibrium state is globally asymptotically stable if the reproductive numberis more than one. Four theorems were proposed as regards the stability of the MERS-Cov transmission using Lyapunov stability theories. Theorems 1 and 2 proved the exponential stability of the MERS-Cov system, while Theorems 3 and 4 proved the global asymptotical stability of the MERS-Cov system.



3(a) 3(b)

3(c) 3(d)

Figure 3. (a) The susceptible population (b) The exposed population (c) The infected population and (d) The recovered population.

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