Mean and Variance Adjustment of the Average Control Chart by Shape Parameter Using Bayesian Estimation of the Inverse Gaussian Distribution

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Abstract

This research aims to develop the average control chart ($\bar{x}-chart$) using the shape parameter of the Inverse Gaussian Distribution by Bayesian Estimation for estimating mean and variance, and to compare the process potential capability (C_p) and the actual process capability index (C_{pk}) for Monte Carlo simulation with $10{,}000$ replications assuming that the specification is ± 0.001. The result shows that the process potential capability (C_p) and the actual process capability index (C_{pk}) of the Adjusted $\bar{x}-chart$ using Bayesian Estimation of the shape parameter of the Inverse Gaussian Distribution for estimating mean and variance have more capability than the $\bar{x}-chart$ under the normal distribution when the sample size is less than 30. For the sample size of 30, the two control charts have the indifferent capability process.

Keywords: Adjusted $\bar{x}-chart$, Bayesian Estimation, shape parameter, Inverse Gaussian Distribution
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1. Introduction

Statistical Process Control is a critical tool in maintaining the quality of products and services in the new manufacturing process to meet the standards that manufacturers and consumers require. It is the highest satisfaction for the products and services in order to maximize profits in the long run. This will result in the company to be able to continue its quality control. Statistical methods are used to calculate and apply the results for decision-making in relation to the quality of products in various areas such as the development of products to meet the standards of the manufacturer itself and the development of product standards to have equivalent level to other manufacturers in the market [1].

The most widely used instrument for statistical quality control is a control chart that applies the attribute data such as p-chart and np-chart to estimate mean and variance under assumption of the data with Binomial distribution. For the variable data, the control charts, like

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$\bar{x}-chart$, R-chart are used. Estimating mean and variance under the assumptions have a normal distribution [2-3].

The attribute control chart has broadly been developed. Quesenberry [4] proposed a Q Chart for binomial random variables for controlling the proportion of waste in the process. Khoo [5] offered a control chart for moving averages. De Oliveira *et al*. [6] and Ryand and Schwertman [7] proposed beta control chart. Their results showed that the Q control chart has more efficiency than the control chart controlling the proportion of waste in the process.

Moreover, Rungruang [8] compared the efficiency of 3 control charts which are beta, moving average and the queue control chart. When the data has a binomial distribution in the parameters n and p, the criteria used to compare the efficiency of the control chart is the average run length (ARL) of 240 simulating situations with replication of 10,000 times by Monte Carlo technique. It was found that when the process was under the moving averages and queue control charts, it was equally efficient without process control. For the beta control charts, it was more efficient in case of less number of waste with small variation. On the other hand, the moving average control chart was more efficient in case of large number of waste with large variation at all levels of sample size.

For the variable data, the widely used control charts are $\bar{x}-chart$, R-chart to estimate mean and variance under the assumption of normal distribution. If the data is not normal distribution, the normal distribution in estimating mean and variance for controlling in the upper limit (UCL) and lower limit (LCL) will lead to high error of the estimations. So, we apply the shape parameter (α) of the Inverse Gaussian Distribution by Bayesian estimation for adjusted mean and variation in average control chart ($\bar{x}-chart$). Moreover, we compare the process potential capability (C_p) and the actual process capability index (C_{pk}) for Monte Carlo simulation with 10,000 replications.

2. Materials and Methods

The Inverse Gaussian Distribution has right distribution (Figure 1). The probability density function is:

$$f_{IG}(x:\alpha,\beta)=\sqrt{\frac{\beta}{2\pi}}x^{-\frac{3}{2}}\exp\{\frac{\beta(x-\alpha)^2}{2\alpha^2 x}\},\ x>0 \tag{1}$$

where α is the shape parameter ($E(X)=\alpha$) and β is the scale parameter [9].

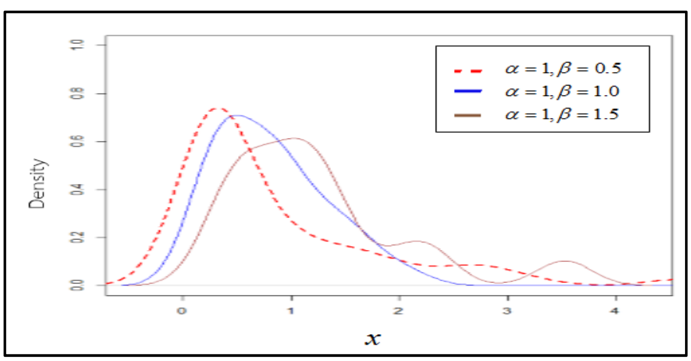

Figure 1 Probability density function of the Inverse Gaussian

The estimation of the shape parameter (α) of the Inverse Gaussian Distribution by Bayesian estimation using Weibull as prior distribution can be derived as:

$$h_{WB}(\alpha,\beta \mid x_1,x_2,...,x_n) = \frac{L_{IG}(x:\alpha,\beta)g_{WB}(\alpha,\beta)}{\int_0^\infty\int_0^\infty L_{IG}(x:\alpha,\beta)g_{WB}(\alpha,\beta)d\alpha d\beta} \tag{2}$$

Where $h_{WB}(\alpha,\beta \mid x_1,x_2,...,x_n)$ is posterior distribution,

$L_{IG}(x:\alpha,\beta)$ is likelihood function,

$g(\alpha,\beta)$ is Weibull prior distribution, The probability density function is as:

$$g_{WB}(\alpha,\beta) = \alpha\beta x^{\alpha-1}e^{\{-\beta x^{\alpha}\}} \ , \ x>0\,,\alpha>0,\beta>0 \tag{3}$$

The expectation of the parameter α is as:

$$\hat{\alpha}_{WB} = E(\alpha \mid x_1,...,x_n) \tag{4}$$

$$\hat{\alpha}_{WB} = \int_0^\infty \alpha.h_{WB}(\alpha,\beta \mid x_1,x_2,...,x_n)d\alpha \text{ [10].} \tag{5}$$

The (5) equation is not integral in close form so we cannot find the posterior distribution. We use Lindley's Approximation [11] within square error loss, the equation for approximation is as:

$$\hat{\alpha}_{WB} = \hat{\alpha}_{MLE} + \frac{1}{2}(\theta_2 + 2\theta_1\rho_1)\sigma^2 + \frac{1}{2}l_3\theta_1\sigma^4 \tag{6}$$

Where θ_1 is the first Derivative of α by α

θ_2 is the second Derivative of α by α

ρ is natural Logarithm of Weibull Prior Distribution

ρ_1 is the first Derivative of ρ by α

l is natural Logarithm of Inverse Gaussian Distribution Function by α

l_2 is the second Derivative of l Function by α

l_3 is the third Derivative of l Function by α

$\sigma^2 = \frac{(-1)}{l_2}$,

So

$$\hat{\alpha}_{WB} \approx \hat{\alpha}_{MLE} + \rho_1\sigma^2 + \frac{1}{2}l_3\sigma^4\hat{\alpha}_{WB} \tag{7}$$

$$\hat{\alpha}_{WB} \approx \hat{\alpha}_{MLE} + \left(\frac{1}{\frac{3\hat{\beta}_{MLE}\sum_{i=1}^{n} x_i}{(\hat{\alpha}_{MLE})^4} - \frac{2n\hat{\beta}_{MLE}}{(\hat{\alpha}_{MLE})^3}}\right)\left(\left(\frac{n}{\hat{\alpha}_{MLE}} + \sum_{i=1}^{n} \ln x_i - \hat{\beta}_{MLE}\sum_{i=1}^{n} x^{\alpha} \ln x_i\right) + \left(\frac{6\hat{\beta}_{MLE}\sum_{i=1}^{n} x_i}{(\hat{\alpha}_{MLE})^5} - \frac{3n\hat{\beta}_{MLE}}{(\hat{\alpha}_{MLE})^4}\right)\left(\frac{1}{\frac{3\hat{\beta}_{MLE}\sum_{i=1}^{n} x_i}{(\hat{\alpha}_{MLE})^4} - \frac{2n\hat{\beta}_{MLE}}{(\hat{\alpha}_{MLE})^3}}\right)\right) \quad (8)$$

where $\hat{\alpha}_{MLE} = \overline{X}$ and $\hat{\beta}_{MLE} = \frac{n}{\sum_{i=1}^{n}\left(\frac{1}{X_i} - \frac{1}{\overline{\overline{X}}}\right)}$ [12]

The variance of the shape parameter (α) has the following formula

$$Var(\hat{\alpha}_{WB}) = \frac{\sum_{i=1}^{n}(\hat{\alpha}_{(WB)_i} - \overline{\alpha}_{WB})^2}{n-1} \quad (9)$$

We obtained the estimator of mean and variance of the shape parameter in the Inverse Gaussian Distribution. We adjusted mean and variance of the formula of average control chart ($\overline{x}-chart$ **).** The formula of $\overline{x}-chart$ is as:

$$UCL = \overline{\overline{X}} + 3\frac{\sigma}{\sqrt{n}} \quad (10)$$

$$CL = \overline{\overline{X}} \quad (11)$$

$$LCL = \overline{\overline{X}} - 3\frac{\sigma}{\sqrt{n}} \quad (12)$$

When $\overline{\overline{X}} = \frac{\sum_{i=1}^{n} X_i}{n}$ and $\sigma = \sqrt{\frac{\sum_{i=1}^{N}(X-\mu)^2}{N}}$

Then, we replaced $\overline{\overline{X}} = \hat{\alpha}_{WB}$ and $\sigma = \sqrt{Var(\hat{\alpha}_{WB})}$

The formula of the adjusted $\overline{x}-chart$ is as :

$$UCL = \hat{\alpha}_{WB} + 3\frac{\sqrt{Var(\hat{\alpha}_{WB})}}{\sqrt{n}} \quad (13)$$

$$CL = \hat{\alpha}_{WB} \quad (14)$$

$$LCL = \hat{\alpha}_{WB} - 3\frac{\sqrt{Var(\hat{\alpha}_{WB})}}{\sqrt{n}} \tag{15}$$

So, we calculate the process potential capability (C_p) and actual process capability index (C_{pk}) with in assume that the specification is ± 0.001. The formula of the process potential capability (C_p) is as :

$$C_p = \frac{USL - LSL}{UCL - LCL} \tag{16}$$

where USL = upper specification limit

LSL = *lower* specification limit

For the actual process capability index (C_{pk}) the formula is as :

$$C_{pk} = \min(C_{pu}, C_{pl}) \tag{17}$$

Where $C_{pu} = \dfrac{USL - \overline{\overline{X}}}{3\sigma}$,

$C_{pl} = \dfrac{\overline{\overline{X}} - LSL}{3\sigma}$ [13] .

C_p = Process Capability. A simple and straightforward indicator of process capability.
C_{pk} = Process Capability Index. Adjustment of C_p for the effect of non-centered distribution.

Afterwards, simulation in R program was implemented supposing that $\alpha = 0.01, 0.02, 0.03, 0.04, 0.05$ (the Copper plating process in Gravure Printing was stopped when the values exceed), $n = 5, 10, 15, 20, 25, 30$ for 10,000 replications and calculating the process potential capability (C_p) and the actual process capability index (C_{pk}), assuming that specification is ± 0.001 because this can be accepted by customers. The criteria of C_p and C_{pk} are not less than 1 for the process ability to be accepted (ISO/TS 16949).

3. Results

The simulation was computed by Monte Carlo from 30 situations (α = 0.01, 0.02, 0.03, 0.04, 0.05 and n = 5, 10, 15, 20, 25, 30) with R Program of 10,000 replications in specification ± 0.001. Furthermore, the process potential capability (C_p) and the actual process capability index (C_{pk}) of the average control chart ($\bar{x}-chart$) under normal distribution and adjusted average control chart (Adjusted $\overline{x}-chart$) under Inverse Gaussian Distribution with shape parameter (α) were calculated. The results were as follows:

In case of $n=5$, the Adjusted $\overline{x}-chart$ has more capability in process than the $\overline{x}-chart$ in customer requirement in parameter $\alpha=0.04$ ($C_p=1.63$, $C_{pk}=1.47$) and $\alpha=0.05$ ($C_p=1.76$, $C_{pk}=1.22$) but the $\overline{x}-chart$ has no capability in process for customer requirement.

In case of $n=10$, the Adjusted $\overline{x}-chart$ has more capability in process than the $\overline{x}-chart$ in customer requirement in parameter $\alpha=0.01, 0.02, 0.03, 0.04, 0.05$ (all situations) and the $\overline{x}-chart$ has capability in process for customer requirement in parameter $\alpha=0.01$ ($C_p=1.53$, $C_{pk}=1.25$) but the other case of the $\overline{x}-chart$ has no capability in process for customer requirement.

In case of $n=15$, the Adjusted $\overline{x}-chart$ has more capability in process than the $\overline{x}-chart$ for customer requirement in parameter $\alpha=0.01$ ($C_p=1.30$, $C_{pk}=1.23$), $\alpha=0.02$ ($C_p=1.45$, $C_{pk}=1.32$) and $\alpha=0.03$ ($C_p=1.73$, $C_{pk}=1.34$) but the $\overline{x}-chart$ has capability in process for customer requirement in parameter $\alpha=0.01$ $C_p=1.25$, $C_{pk}=1.13$) and $\alpha=0.02$ ($C_p=1.78$, $C_{pk}=1.56$).

In case of $n=20$, the Adjusted $\overline{x}-chart$ has more capability in process than the $\overline{x}-chart$ for customer requirement in parameter $\alpha=0.01, 0.02, 0.03, 0.04, 0.05$ (all situations), but the $\overline{x}-chart$ has capability in process for customer requirement in parameter $\alpha=0.01$ ($C_p=1.17$, $C_{pk}=1.01$), $\alpha=0.03$ ($C_p=1.79$, $C_{pk}=1.35$) and $\alpha=0.04$ ($C_p=1.87$, $C_{pk}=1.42$).

In case of $n=25$, the Adjusted $\overline{x}-chart$ has more capability in process than the $\overline{x}-chart$ for customer requirement in parameter $\alpha=0.01$ ($C_p=1.70$, $C_{pk}=1.35$) and $\alpha=0.02$ ($C_p=1.76$, $C_{pk}=1.36$) but the $\overline{x}-chart$ has capability in process for customer requirement in parameter $\alpha=0.04$ ($C_p=1.80$, $C_{pk}=1.55$) and $\alpha=0.05$ ($C_p=1.89$, $C_{pk}=1.40$).

In case of $n=30$, the Adjusted $\overline{x}-chart$ and the $\overline{x}-chart$ have capability in process for customer requirement in all situations (Table 1 and Figure 2).

Table 1. Situations for comparing C_p, C_{pk} in $\bar{x}-chart$ and Adjusted $\bar{x}-chart$.

Situations		Type of Control Chart			
		$\bar{x}-chart$		Adjusted $\bar{x}-chart$	
n	α	C_p	C_{pk}	C_p	C_{pk}
5	0.01	0.85	0.72	0.82	0.97
	0.02	0.87	0.73	0.78	0.88
	0.03	0.75	0.66	0.78	0.67
	0.04	0.93	0.85	1.63**	1.47**
	0.05	0.98	0.86	1.76**	1.22**
10	0.01	1.53**	1.25**	1.84**	1.47**
	0.02	0.98	0.87	1.75**	1.39**
	0.03	0.79	0.75	1.85**	1.42**
	0.04	0.87	0.82	1.77**	1.32**
	0.05	0.98	0.85	1.89**	1.38**
15	0.01	1.25**	1.13**	1.30**	1.23**
	0.02	1.78**	1.56**	1.45**	1.32**
	0.03	0.99	0.92	1.73**	1.34**
	0.04	0.87	0.75	0.85	0.76
	0.05	0.78	0.63	0.94	0.75
20	0.01	1.17**	1.01**	1.87**	1.47**
	0.02	0.96	0.88	1.95**	1.52**
	0.03	1.79**	1.35**	1.89**	1.49**
	0.04	1.87**	1.42**	1.67**	1.34**
	0.05	1.88**	1.39**	1.60**	1.32**
25	0.01	0.68	0.40	1.70**	1.35**
	0.02	0.76	0.55	1.76**	1.36**
	0.03	0.79	0.60	0.89	0.45
	0.04	1.80**	1.55**	0.93	0.56
	0.05	1.89**	1.40**	0.78	0.44
30	0.01	1.87**	1.45**	1.79**	1.45**
	0.02	1.95**	1.37**	1.70**	1.44**
	0.03	1.88**	1.36**	1.95**	1.67**
	0.04	1.75**	1.32**	1.99**	1.70**
	0.05	1.78**	1.36**	1.59**	1.23**

** $C_p, C_{pk} > 1$, the capability in process is in customer requirement (ISO/TS 16949)

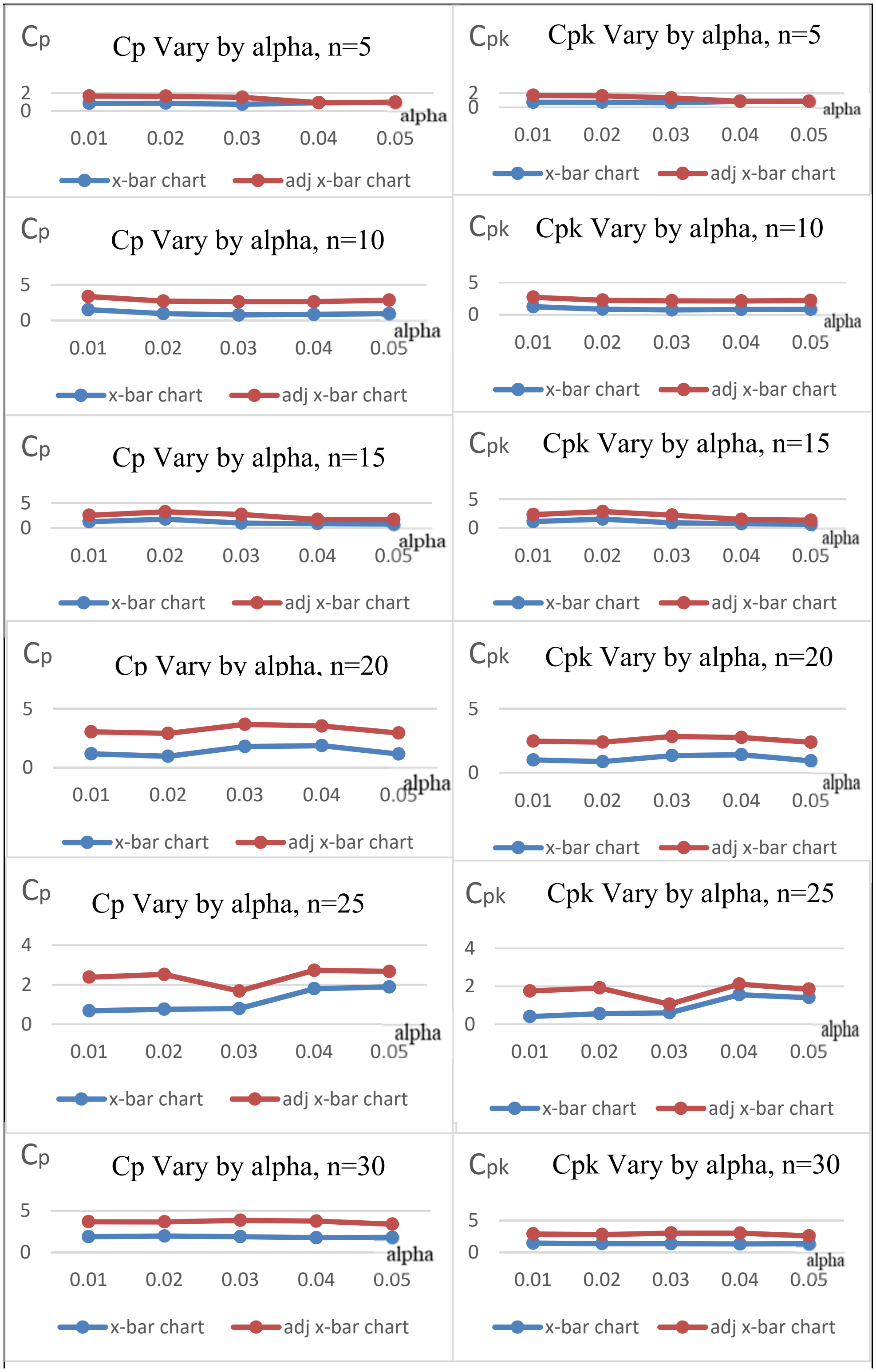

Figure 2. C_p, C_{pk}, $\alpha = 0.01, 0.02, 0.03, 0.04, 0.05$ and $n = 5, 10, 15, 20, 25, 30$

4. Conclusions

The process potential capability (C_p) and the actual process capability index (C_{pk}) of the average control chart ($\bar{x}-chart$) and the Adjusted control chart (Adjusted $\bar{x}-chart$) show that when the sample size is 5-25, the process potential capability (C_p) and actual process capability index (C_{pk}) of the Adjusted control chart (Adjusted $\bar{x}-chart$) have more capability in process for customer requirement than the average control chart ($\bar{x}-chart$). Moreover, when the sample size is 30, the process for customer requirement is capable of both control charts. For the shape parameter supposing that the simulation for Copper plating process in Gravure Printing (α = 0.01, 0.02, 0.03, 0.04, 0.05), it was found that the parameter is small. Further investigation shoud be done by Adjusted $\bar{x}-chart$ in situations of the big shape parameter. For the shape parameter (α), the Bayesian estimated in the Inverse Gaussian Distribution using the Weibull as prior distribution was more efficient when the sample size was small. The approximation of the shape parameters is impossible to find the formulas in the form of closed form. Therefore, Lindley's Approximation technique is consistent [10]. It can be concluded that the shape parameter of the Inverse Gaussian using Weibull as prior distribution has more efficiency than using Gamma as prior distribution in case of small sample size.

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