Two-Dimensional Cutting Stock Problems with a Modified Column Generation Method

Sirirat Juttijudata* and Phissanu Sudjarittham

Applied Optimization Research Unit, Faculty of Engineering Sriracha,
Kasetsart University, Sriracha Campus, Chonburi, Thailand

Received: 17 January 2020, Revised: 6 February 2020, Accepted: 5 March 2020

Abstract

The two-dimensional cutting stock problems pose mathematical challenges due to the nature of mixed integer linear programming resulting in NP-hard problems. At the same time, the problems are industrially important in manufacturing, logistic and supply chain industries. The ability to solve large-scale two-dimensional cutting stock problems could have a great impact on research community as well as industries. The objective of this work is to develop a framework of solution method for two-dimensional cutting stock problems using a modified column generation method. Two-stage Guillotine cutting patterns are considered. The relationship between the cutting patterns in the first and second stages gives rise to additional constraints that are not previously found in one-dimensional cutting stock problems. As a result, the column generation method was modified to handle these additional constraints. In order to further simplify the problem, LP relaxation is used in conjunction with the column generation. Integer solution can be obtained by rounding of LP solution. The lower bound of the problems may be estimated from the minimization of LP problem; allowing the optimality of solution obtained to be assessed in terms of, for example, the worst performance ratio. With the instance problem studied in this work, the modified column generation method performs well and produces the optimal result that is only 1% less than optimal solution obtained from the exact algorithm, which is the effect of rounding. In terms of speed, the proposed method requires only 1/200 floating point operations compared to the full problem with all feasible solutions (from the instance problem studied here). The proposed method may be further fine-tuned both in terms of rounding techniques with some tweaks in the column generation method in the future. The special structures of the problems should be further exploited for the advantage of the solution methods for large-scale problems.

Keywords: two-dimensional cutting stock problems, Guillotine constraints, column generation
DOI 10.14456/cast.2020.11

*Corresponding author: Tel.: +66 16 46 5636
E-mail: srcsrr@ku.ac.th

1. Introduction

There is a wide variety of industrial applications of the two-dimensional cutting stock problems such as those in wood, glass and paper industries. The goal of a two-dimensional cutting stock problems is to minimize the number of standard-size stock sheets to be cut into pieces of specified sizes. In general, the standard-size stock sheets can be cut into either regular (rectangle, circular, etc.) or irregular shapes depending on the applications. The mathematical model and solution procedures of the cutting-stock problems can also be applied in other problems such as bin-packing, knapsack, vehicle loading, pallet loading and car loading problems. According to Delorme *et al*. [1], the number of articles having the titles related to either "cutting stock problems," "bin-packing problems," or both increased from approximately 30 articles per year before 2000 to 130 articles per year in 2014 in Google Scholar database.

For the two-dimensional rectangular guillotine cutting stock problems considered in this study (further explained in the next section), Gilmore and Gomory in 1965 [2] proposed the solution procedure based on an integer programming model with a cutting pattern (column) generation procedure by solving two one-dimensional knapsack problems. In 1966, Gilmore and Gomory [3] further elaborated the cutting pattern generation procedure based on dynamic programming recursions. Since the cutting-stock problems are NP-hard problems, it becomes more complex to generate all feasible cutting patterns as the size of problem increased, which is typically found in practice. Heuristic and metaheuristic procedures represent more practical approaches, e.g., the partial enumeration heuristics of all feasible patterns by Benati in 1997 [4], the linear programming (LP) relaxation with the rounding procedure by Johnson in 1986 [5], the three-stage-sequential heuristics by Suliman in 2005 [6], genetic algorithms by Hopper and Turton in 1999 [7], and the ant colony optimization by Levine and Ducatelle in 2004 [8].

Even though the heuristic and metaheuristic approaches could alleviate some of our difficulties in terms of the complexity of large-scale cutting stock problems in practice, they are still not an absolute mean to solve such problems. Furthermore, there is no guarantee that the solution obtained will be optimal or at least close to optimal. Because of this gap, the objective of this paper is to develop a solution procedure for large-scale two-dimensional rectangular guillotine cutting stock problems. The method should also provide the lower bound as a gauge to check the optimality of the solution. Consequently, this study proposes the solution procedure based on the LP relaxation i.e. the column generation and the rounding.

2. Materials and Methods

2.1 Two-dimensional (2D) cutting stock problem

2.1.1 Mathematical model

For two-dimensional cutting stock problems, the standard-size stock sheets are cut into pieces of specified sizes subjected to 2 stage guillotine constraint-uninterrupted cuts going from one end to the opposite end of the sheet in two perpendicular directions sequentially as shown in Figure 1 while minimizing the number of standard-size stock sheets used. Gilmore and Gomory [2] developed the mathematical model for n-stage cutting stock problems of two and more dimensions. Their 2 stage two-dimensional cutting stock (with trimming) mathematical model is adopted in this paper. In the first stage cut, the standard size sheet of $W \times L$ will be cut into strips with given cutting patterns shown in Figure 1.

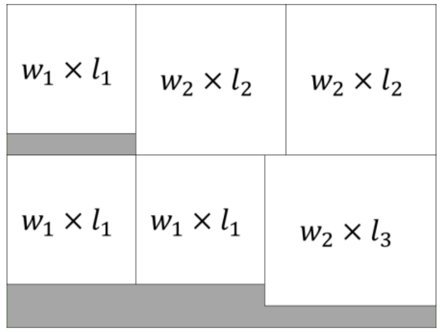


Figure 1. The 2-stage guillotine cutting pattern with trimming wasted material shown in the rendered area. The standard size sheet was $W \times L$ and pieces with sizes $w_i \times l_j$

The number of total strips N_{w_i} with w_i width obtained from the first stage cut will then be:

$$N_{w_i} = \sum_{p=1}^{P} a_{ip} x_p$$

where x_p was the number of standard sheets cut with p pattern at stage 1. The p cutting pattern a_{ip} is defined as the number of strips w_i cut along the width of standard size. Note that i is the index for the widths of strips running from $i = 1, \ldots, I$, where I represents the total number of widths of strips. Figure 1 showed the strips of sizes $w_1 = 6$ units and $w_2 = 7$ units, i.e. $I = 2$. If $p = 1$ in Figure 1, the cutting pattern in the first stage for strips w_1 and w_2 is:

$$a_{i1} = \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}.$$

Note that the summation of the strip widths in the cutting pattern must be less than W, i.e.

$$\sum_{i=1}^{I} w_i a_{ip} \leq W, \forall p. \tag{1}$$

In the second stage, the cutting will be performed in the perpendicular direction, i.e. the vertical direction along the w_i strips with q cutting pattern b_{ijq}. Noted that j is the index of piece lengths running from $j = 1, \ldots, J$. For w_2 strips in Figure 1, the cutting pattern $q = 1, 2$ for the pieces with l_1, l_2 and l_3, respectively, are:

$$b_{2j1} = \begin{Bmatrix} 1 \\ 2 \\ 0 \end{Bmatrix} \text{ and } b_{2j2} = \begin{Bmatrix} 2 \\ 0 \\ 1 \end{Bmatrix}.$$

Remember that the strip with w_i width can also be trimmed in to narrower pieces as illustrated in Figure 1. Constraint (1) must also be satisfied along the length of the strip w_i :

$$\sum_{j=1}^{J} l_j b_{ijq} \le L, \forall i, q. \quad (2)$$

As the total number of strip w_i is limited by N_{w_i} from the first stage, the relationship between the number of strips w_i, y_q, and x_p is related via:

$$\sum_{q=1}^{Q} y_q \le \sum_{p=1}^{P} a_{ip} x_p , \forall i. \quad (3)$$

After the cutting process in both stages, the number of pieces $w_i \times l_j$ must be at least equal to the demand of such pieces d_{ij} :

$$\sum_{q=1}^{Q} b_{ijq} y_q \ge d_{ij}, \forall i, j \quad (4)$$

with the objective to minimize the total number of standard-size stock sheet used in the cutting process:

$$\min z = \sum_{p=1}^{P} x_p . \quad (5)$$

In summary, the mathematical model for a two-stage cutting stock problem in two-dimensions is:

$$\min z = \sum_{p=1}^{P} x_p .$$

Subjected to

$$\sum_{p=1}^{P} a_{ip} x_p - \sum_{q=1}^{Q} y_q \ge 0, \forall i$$

$$\sum_{q=1}^{Q} b_{ijq} y_q \ge d_{ij}, \forall i, j$$

$$x_p, y_q \ge 0$$

$$x_p, y_q \in I,$$

where $\sum_{i=1}^{I} w_i a_{ip} \le W, \forall p$ and $\sum_{j=1}^{J} l_j b_{ijq} \le L, \forall i, q.$

2.1.2 Instance problem

Let us consider the instance problem of 2-stage cutting stock in two-dimension with the standard-size stock sheets of width W and length L of 15 and 20 units, respectively. The demands d_{ij} for pieces with size of $(w, l) = (6,6), (6,7), (6,8), (7,6)$, (7,7) and (7,8) are 30, 15, 5, 5, 15 and 25, respectively. The problem instance is summarized in Table 1.

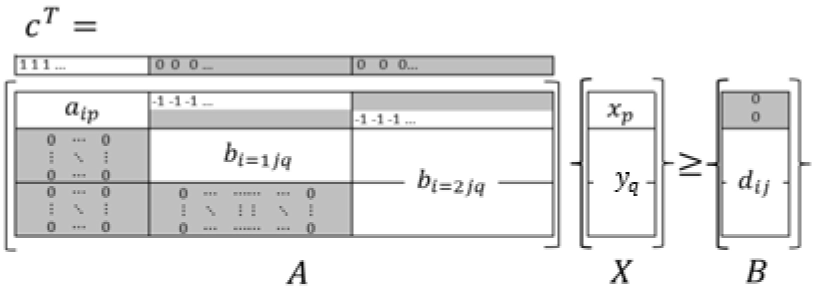
Table 1. The summary of problem instance

Standard-size stock sheet		
W L		15 20
Demand size and demand		
The width of the strip	*The length of the piece*	*Demand*
$w_1 = 6$	$l_1 = 6$ $l_2 = 7$ $l_3 = 8$	30 15 5
$w_2 = 7$	$l_1 = 6$ $l_2 = 7$ $l_3 = 8$	5 15 25

The mathematical model of the problem with all feasible cutting patterns following constraints (1), (2) is formulated in Microsoft Excel 2013 in the matrix form as:

$$\min z = c^T X$$
$$A\,X \leq B$$
$$X \geq 0 \text{ } and \text{ } X \in I$$

Where:



All feasible cutting patterns a_{ip} and b_{ijq} are generated by means of the search tree resulting in the number of cutting patterns for $a_{ip}, P = 5$ and the number of cutting patterns for $q_{1jq} + q_{2jq}, Q = 13 + 49 = 62$. Therefore, the total number of decision variables x_p and y_q, and constraints are 67 and 8, respectively. The optimal solution was determined by Microsoft Excel Solver with Simplex LP, constraint precision of 1×10^{-6}, automatic scaling, and integer optimality of 1%. For the optimal solution, the total number of standard-size stock sheets needed to be cut to meet the demands exactly of 175. The cutting patterns in the first stage a_{ip} and x_p for the optimal solution are:

$$a_{ip} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \text{ with } x_p = \begin{Bmatrix} 75 \\ 100 \end{Bmatrix}$$

and the cutting patterns in the second stage b_{ijq} and y_q for the optimal solution are as follows:

Strip $w_1 = 6$:

$$b_{1jq} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ with } \left(y_q\right)_1 = \begin{Bmatrix} 50 \\ 50 \end{Bmatrix}$$

and strip $w_2 = 7$:

$$b_{2jq} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \text{ with } \left(y_q\right)_2 = \begin{Bmatrix} 100 \\ 50 \\ 50 \\ 50 \end{Bmatrix}$$

where the cutting patterns in the upper section of b_{2jq} for strips $w_2 = 7$ corresponds to the cutting patterns for the strips with $w_1 = 6$ while the lower section corresponds to those for the strips with $w_2 = 7$. Remember with the strips of $w_2 = 7$, pieces with the width of either $w_1 = 6$ or $w_2 = 7$ can be cut from them.

2.2 Column generation method

2.2.1 One-dimensional (1D) cutting stock problems

As the size of cutting stock problems increase, the number of all feasible cutting patterns grows rapidly. It becomes impractical, even impossible, to include all the cutting patterns into the problems even it is a one-dimensional cutting stock problem. Instead, the column generation algorithm is used to solve the problem. In column generation algorithm [9], the problem is formulated as a restricted master problem (RMP) with as few decision variables (as well as cutting patterns) as possible, i.e. the model with $P \subset P_{all}$ and $Q \subset Q_{all}$. The new decision variables and cutting patterns are brought into the basis as needed in a similar manner to the simplex method via the following sub-problem:

$$\min RCC = \min\left(1 - \sum_{i=1}^{I} \pi_i a_{ip}\right) = 1 - \max\left(\sum_{i=1}^{I} \pi_i a_{ip}\right)$$

Subjected to

$$\sum_{i=1}^{I} w_i a_{ip} \leq W, \forall p$$

$$a_{ip} \in I, \forall i, p.$$

RCC and π_i are a reduced cost coefficient and a shadow price from the restricted master problem. This sub-problem is indeed a knapsack problem which has been studied extensively and may be solved efficiently by many algorithms, e.g., dynamic programming. As in the simplex method, the criterion for possible improvement is $\min RCC < 0$ or $\max\left(\sum_{i=1}^{I} \pi_i a_{ip}\right) > 1$ and the corresponding column a_{ip} , i.e. the cutting pattern will be enter the master problem as the basis and at the optimal solution, $\min RCC \geq 0$.

2.2.2 Two-dimensional (2D) cutting stock problems

For two-dimensional cutting stock problems, the Guillotine constraint and relationship between x_p and y_q impose certain constraints on column a_{ip} and $b_{ijq}$of the restricted master problem as seen in the matrix form of the instance problem discussed above. Two modifications are proposed here to handle the two-dimensional cutting stock problems.

First, as the structures of columns of matrix A are having different structures, the sub-problem for each structure shall be formulated individually. Let us take the example of the instance problem. There are three different structures for columns of A in response to the cutting patterns for the first stage, those for the second stage with the strips of w_1 width, and those for the second stage with the strips of w_2; there will be three different formulations for the Knapsack sub-problems. The decision variables for sub-problems are only the cutting patterns a_{ip} and b_{ijq} while constants 0 and -1 in those columns are treated as a constant contribution in sub-problems.

The second modification comes from the fact that there will be three columns generated at each stage rather than just one column, in the case of the instance problem discussed above; thereby three newly generated columns will be added as the bases to the problem concurrently at the end of each iteration.

The solution of master and sub-problem continues until the optimal solution has arrived - either when $RCC \geq 0$ or there are no new independent columns generated.

It should be noted that to get the shadow price for the knapsack problem, the (restricted) master problem has to be relaxed to LP problem - the integer constraint was removed from the problem. As a result, the LP solution obtained may not necessary be integer and required further rounding procedure to round them into integer. Elaborate methods such as branch-and-bound may be used but as the preliminary study of the extension of column generation algorithm for two-dimensional cutting stock problems, simply rounding to the nearest integer is adopted at the end of optimal solution from relaxed LP problem.

The optimal solution of LP problem may also be used as a lower bound to evaluate the worst performance ratio $WCPR = z^*/z_{LP}$ of the solution procedure.

3. Results and Discussion

The modified column generation technique was applied to the instance problem in section 2.1.2. The procedure started from as few decision variables as possible but sufficient to generate feasible solutions. Master and sub-problems were solved sequentially over and over until the optimal solution converged. Among some different initial conditions experimented in this study, the column generation algorithm took less than 10 iterations to arrive at the relaxed optimal solution. The LP optimal solutions for all the cases are 175 coincide well with the exact solution in section 2.1.2. The optimal solutions with cutting patterns, x_p and y_q are:

$$a_{ip} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ with } x_p = \begin{Bmatrix} 62.5 \\ 112.5 \end{Bmatrix}$$

$$b_{1jq} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \text{ with } \left(y_q\right)_1 = \begin{Bmatrix} 75 \\ 50 \end{Bmatrix}$$

$$b_{2jq} = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \text{ with } \left(y_q\right)_2 = \begin{Bmatrix} 75 \\ 25 \\ 25 \\ 100 \end{Bmatrix}$$

With the rounding, the (sub) optimal solution from the solution procedure is 176. With the worst performance ratio = 176/175 = 1.006, it is clear that the solution from the procedure is only 0.6% higher than the lower bound, i.e. the LP solution, which is typically lower than the integer programming model; the solution procedure produces satisfactory result. More elaborate rounding procedure is necessary to improve our method and shall be studied in the future.

The column generation method proposed here only adds four more cutting patterns b_{ijq} (columns), i.e. adding four new decision variables to the problem. In total, there are only 12 decision variables to be considered in the column generation procedure. Comparing the number of decision variables to those from the exact LP problem in section 2.1.2, the column generation method only deals with 12 decision variables or only about 20% of the full problem resulting in almost 200 times faster in terms of SIMPLEX floating point operations. Moreover, with the column generation method, it is not necessary to generate all feasible cutting patterns which can be rather challenging for large-scale problem.

Along the course of solution determination, it is observed that at the beginning, the columns corresponding to the cutting patterns in the first stage of cutting process a_{ip} are generated and dominated the whole process. This suggests the method tried to generate stocks of strips with different widths as the resource of the second-stage cutting process. Later on, columns in response to the cutting patterns in the second stage are added. Most of these cutting patterns are for wider strips, w_2, with cutting patterns for pieces with mixture of $w_1$and w_2. This should be the result that the wider strips are more flexible in terms of patterns to be cut; the solution is driven into that direction.

4. Conclusions

The main contribution of this paper is to develop an alternative algorithm to solve large-scale two-cutting stock problems based on the column generation with LP relaxation. The mathematical models of such problems are derived from Gilmore and Gomory [2]. Guillotine constraint and the relationship between x_p and y_q impose certain constraints on columns a_{ip} and b_{ijq}; hence the column generation algorithm has to be modified to cope with the change. The restricted master problems (relaxed LP) and knapsack sub-problems were solved sequentially until the optimal solution are reached. Rounding was applied at the end of the process resulting in slightly less than 1% optimal solutions compared to the full exact solution. The lower bound of the optimal solution may be estimated from z^* of LP. Consequently, the worse-performance ratio can be evaluated and the optimality of the obtained solution can be gauged. It should be noted that in terms of floating point operations, the column generation method requires less 1/200 of those required by exact solution. In other words, the column generation method is 200 times faster than the full exact solution.

There are several aspects that can be done in the future: to refine the column generation method, to improve the rounding technique such as branch-and-bound method, and to exploit the special structures of problems for large-scale problems. Further comparison with other solution procedure shall be carried out to benchmarking the performance of the proposed method.

References

[1] Delorme, M., Iori, M. and Martello, S., 2016. Bin packing and cutting stock problems: mathematical models and exact algorithms, *European Journal of Operational Research*, 225 (1), 1-20.

[2] Gilmore , P.C. and Gomory, R.E., 1965. Multistage cutting stock problems of two and more dimensions, *Operations Research*, 13, 94-120.

[3] Gilmore , P.C. and Gomory, R.E., 1965. The theory and computation of knapsack functions, *Operations Research*, 14, 1045-1074.

[4] Benati, S., 1997. An algorithm for a cutting stock problem on a strip. *Journal of the Operational Research Society*, 39, 288-294.

[5] Johnson, R.E., 1986. Rounding algorithms for cutting stock problems. *Asia-Pacific Journal of Operational Research*, 3, 166-171.

[6] Suliman, S.M.A., 2005. A sequential heuristic procedure for the two-dimensional cutting stock problem. *International Journal of Production Economics*, 99 (1-2), 177-185.

[7] Hopper, E. and Turton, B., 1999. A genetic algorithm for a 2D industrial packing problem. *Computers and Industrial Engineering*, 37 (1-2), 375-378.

[8] Levine, J. and Ducatelle, F., 2004. Ant colony optimization and local search for bin packing and cutting stock problems. *Journal of Operational Research Society*, 55(7), 705-716.

[9] Lasdon, L.S., 1979. *Optimization Theory for Large Systems*. London: MacMillan Publishing.