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Research article

Optimal Control Analysis of Three Control Factors on Susceptible and Infected Compartments for Computer Viruses in a Computer Network

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Abstract

Keywords

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Computer viruses can cause significant damage to computer systems, and that damage can lead to loss of data and financial losses for computer users. To deal with computer viruses and to avoid them in the future, system can be updated and better antivirus software installed. Someone experts can remove viruses without antivirus software and fix infected computers that have serious and great damage. In this paper, we consider three types of control to deal with infected systems and preventing further spread of viruses: the installation of antivirus software on infected computers, the installation of antivirus software on susceptible computers, and the cleaning and repairing of infected computers without the use of antivirus software. We proposed a model that had three equilibrium points: two virus-free and one endemic. Pontryagin's maximum principle was used to solve the problem of optimal control in our model. Some numerical simulations showed that an acceleration in the declining number of infected computers can be achieved by giving control factors on susceptible and infected computers. Furthermore, an increase in relative weights will result in fewer control factors and vice versa.

1. Introduction

Computer technology has developed very rapidly in this era of globalization. The development of software in the modern world is getting more sophisticated from time to time. Along with the rapid development of computers and the use of computer technology as a human aid, the problems faced by computer users are also increasing. Some of these problems are caused by computer viruses. The high dependence on computer performance makes computer viruses a serious threat [1]. Computer viruses, as a type of electronic infection, can cause damage to the computer system they attack.

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Computer viruses can remove, hide, and even change data on a computer system, and they can cause huge financial losses. Antivirus is needed as a solution to prevent the computer virus transmission. Control of computer virus epidemics is a way to mitigate financial impact.

The term "computer virus" is actually taken from the biology term. A computer virus is like a microorganism that can reproduce by transmitting itself to other organisms. There is a similarity between the mechanism of the computer virus transmission to other computer programs and the spread of biological viruses into living cells [2]. Like biological viruses, computer viruses have the ability to transmit, duplicate, and spread by inserting themselves into other computer programs.

An understanding of the dynamics of computer virus transmission can be observed through mathematical models (see [3-6]). Hu *et al*. [3] studied a model for virus propagation between computers in latent period and removable devices, while Gan *et al*. [4] proposed a computer virus propagation model with a complex-network approach. Chen *et al.* [5] studied a delayed SLBS computer virus model. They showed that the optimal control of this model was effective for reducing the number of breakout computers. Yang and Yang [6] studied the effects of removable storage media in their epidemic model. The model has only a unique endemic equilibrium; it has no virus-free equilibrium.

Mathematical models can also be used to predict the number of computers that have been infected by viruses. This is done in order to determine strategies for reducing computer virus transmission. In this research, a mathematical model will be constructed to see the computer virus transmission in a computer network by involving control factors on the computer. The model discussed here is the development of a model studied by Piqueira and Araujo [7], Zhang *et al*. [8], and Qin *et al*. [1]. Our development involved adding a third control on a computer compartment that had not been exposed to a virus. Furthermore, it will be seen that in terms of costs, the addition of the third control is an efficient way of preventing computer virus transmission in a network.

2. Materials and Methods

2.1 Compartmental model for the computer virus transmission in a network system with three control factors

In this section, we consider a mathematical model of computer viruses proposed Zhang *et al*. [8]. We modify the model by involving a third control factor. The model divides the total population into four classes: non-infected computers subjected to possible infection (S), computers which have an active virus (I), computers from which the virus has been removed (R), and antidotal computers equipped with antivirus (A). We assume that the total population, T, with $T = S(t) + I(t) + R(t) + A(t)$, is constant in time. The compartmental model is given in Figure 1 and description of the parameters of the model is given in Table 1.

A susceptible computer may be infected or remain susceptible. Infected computers can transmit the virus through media or web browser. Susceptible computers (S) are infected through contact with infected computers with a rate β. Susceptible and infected computers can be installed with antivirus software and they become antidotal computers with proportion control factor u_3 and u_1, respectively. Infected computers (I), which may remain infected with antivirus, can be

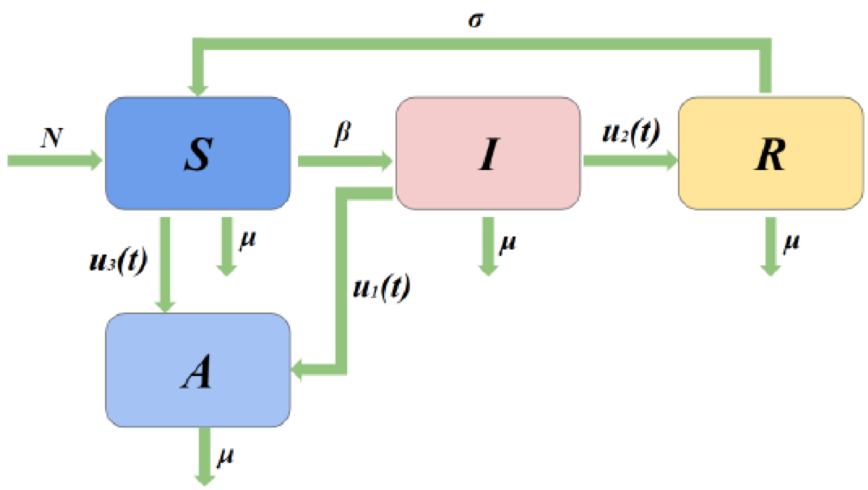

Figure 1. Compartmental model for computer viruses with three control factors. The S, I, R, A represent the numbers of susceptible, infected, removed, and antidotal computer population.

Table 1. Parameters in the model

Symbol of parameters	**Description**	**Unit**
N	Influx rate of new computers to the network	Computer per time
μ	Mortality rate of computer system, not due to the virus	Per time
β	Proportion factor of susceptible into infected because of contact	Per computer per time
σ	Proportion factor of removed into susceptible	Per time
$u_1(t)$	Proportion control factor of infected into antidotal	Per computer per time
$u_2(t)$	Proportion control factor of infected into removed	Per time
$u_3(t)$	Proportion control factor of susceptible into antidotal	Per computer per time

recovered by cleaning and repairing the infected computers without the use of antivirus software with a proportion control factor u_2. Removed computers (R) can be restored by formatting them and they become susceptible computers with a proportion factor σ. All computers removed from the network system (not due to the virus) with a mortality rate μ.

The influx rate of new computers to the network is considered to be $N = 0$ because the virus spreads faster than the network expansion. The same reason justifies the choice of $\mu = 0$, considering that the machine obsolescence time is larger than the time of the virus action. Based on these assumptions, the dynamics of the four computers population with three control factors are expressed mathematically by the following system of four ordinary differential equations (ODEs),

$$\begin{cases} \frac{dS(t)}{dt} = -u_3(t)S(t)A(t) - \beta S(t)I(t) + \sigma R(t) \\ \frac{dI(t)}{dt} = \beta S(t)I(t) - u_1(t)A(t)I(t) - u_2(t)I(t) \\ \frac{dR(t)}{dt} = u_2(t)I(t) - \sigma R(t) \\ \frac{dA(t)}{dt} = u_3(t)S(t)A(t) + u_1(t)A(t)I(t) \end{cases} \tag{1}$$

with the dynamics of the solutions of (1) in the restricted region,

$$\Omega = \{(S, I, R, A) \in \mathbb{R}^4_{+0} | 0 \leq S + I + R + A = T\}$$

and all parameters are real and positive.

A mathematical model has a virus-free equilibrium if it has an equilibrium point at which the population remains in the absence of the virus. The model (1) has two virus-free equilibrium points, given by $E_0(S_0^*, I_0^*, R_0^*, A_0^*) = (0,0,0,T)$ and $E_1(S_1^*, I_1^*, R_1^*, A_1^*) = (T, 0,0,0)$.

System (1) has a unique endemic equilibrium point such that $I(t) > 0$ for any $t > 0$. The endemic equilibrium point is given by:

$$E_2(S_2^*, I_2^*, R_2^*, A_2^*) = \left(p, \frac{T-p}{q+1}, \frac{(T-p)q}{q+1}, 0\right)$$

where $p = \frac{u_2(t)}{\beta}$ and $q = \frac{u_2(t)}{\sigma}$.

To indicate whether the population will remain in the absence of the viruses, or the viruses will persist for all time, we must know the stability of the equilibrium points. Using the theory of stability in epidemiological models, the virus-free equilibrium point E_0 is locally asymptotically stable, while the virus-free equilibrium point E_1 and the endemic equilibrium point E_2 are unstable.

2.2 Optimal control model of computer viruses with three control factors

Now we introduce three control functions $u_i(t), i = 1,2,3$ and three real positive constants $C_i, i = 1,2,3$. The first control u_1 represents a proportion factor of the installation of antivirus on infected computers, the second control u_2 represents a proportion factor of repairing infected computers (they become removed computers), and the third control u_3 represents a proportion factor of the installation of antivirus on susceptible computers. The parameters $C_i > 0, i = 1,2,3$ are appropriate weights of the controls $u_i, i = 1,2,3$ respectively.

We denote the state and control variable of the control system (1) by $x = (S, I, R, A) \in \mathbb{R}^4$ and $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, respectively. For given constant $u_{max} = \Lambda$ and fixed final time $T_f > 0$, the set of admissible control functions is given by

$$U = \{u = (u_1, u_2, u_3): 0 \leq u_i(t) \leq \Lambda, \forall t \in [0, T_f], i = 1,2,3\} \tag{2}$$

The optimal control problem of model for computer viruses with three control factors is formulated as:

[OC] Minimize the functional

$$J(x,u) = \int_0^{T_f} [I(t) + C_1 u_1^2(t) + C_2 u_2^2(t) + C_3 u_3^2(t)]dt \quad (3)$$

subject to the state system (1) and initial conditions

$$S(0) > 0, I(0) \geq 0, R(0) \geq 0, A(0) \geq 0 \quad (4)$$

and control constraints

$$0 \leq u_k(t) \leq \Lambda, \forall t \in [0, T_f] \quad (k = 1,2,3) \quad (5)$$

The objective is a quadratic in control variable $C_i u_i^2(t), i = 1,2,3$ to describe the costs where C_1, C_2, and C_3 are the relative weights attached to the cost of the installation of antivirus software on infected computers, repairing infected computers, and installation of antivirus on susceptible computers respectively.

The goal of the optimal control problem is to find the optimal value u^* of the control function $u = (u_1, u_2, u_3)$ along time that minimizes the cost functional $J(x,u)$ in (3) subject to the dynamic constraints (1), initial conditions (4), and control constraints (5) as well as the number of infected computers at the end of control period.

The Lagrangian of the problem is given by

$$L\big(I(t), u_1(t), u_2(t), u_3(t)\big) = I(t) + C_1 u_1^2(t) + C_2 u_2^2(t) + C_3 u_3^2(t) \quad (6)$$

According to Pontryagin's maximum principle [9], if u^* is optimal for equations (1)-(5) with fixed final time T_f , then there exist adjoint vector $\lambda: [0, T_f] \to \mathbb{R}^4, \lambda = (\lambda_S(t), \lambda_I(t), \lambda_R(t), \lambda_A(t))$, such that

$$S' = \frac{\partial H}{\partial \lambda_S}, I' = \frac{\partial H}{\partial \lambda_I}, R' = \frac{\partial H}{\partial \lambda_R}, A' = \frac{\partial H}{\partial \lambda_A}, \lambda_S' = -\frac{\partial H}{\partial S}, \lambda_I' = -\frac{\partial H}{\partial I}, \lambda_R' = -\frac{\partial H}{\partial R}, \lambda_A' = -\frac{\partial H}{\partial A}$$

where the Hamiltonian for the objective J and the control system (1) as follows

$$\begin{aligned} H(S(t), I(t), R(t), A(t), \lambda(t), u(t)) &= L\big(I(t), u_1(t), u_2(t), u_3(t)\big) + \lambda_S\left(\frac{dS(t)}{dt}\right) + \lambda_I\left(\frac{dI(t)}{dt}\right) + \lambda_R\left(\frac{dR(t)}{dt}\right) \\ &+ \lambda_A\left(\frac{dA(t)}{dt}\right) \\ = I(t) + C_1 u_1^2(t) + C_2 u_2^2(t) + C_3 u_3^2(t) &+ \lambda_S(-u_3(t)S(t)A(t) - \beta S(t)I(t) + \sigma R(t)) \\ &+ \lambda_I(\beta S(t)I(t) - u_1(t)A(t)I(t) - u_2(t)I(t)) + \lambda_R(u_2(t)I(t) - \sigma R(t)) \\ &+ \lambda_A\big(u_3(t)S(t)A(t) + u_1(t)A(t)I(t)\big) \end{aligned}$$

and the minimality condition

$$H(S^*(t), I^*(t), R^*(t), A^*(t), \lambda(t), u^*(t)) = \min_{0 \leq u \leq u_{max}} H(S^*(t), I^*(t), R^*(t), A^*(t), \lambda(t), u(t))$$

holds almost everywhere on $[0, T]$.

We obtain the adjoint equations:

$$\begin{cases} \frac{d\lambda_S}{dt} = -\frac{\partial H}{\partial S} = (u_3(t)A + \beta I)\lambda_S - \beta I\lambda_I - u_3(t)A\lambda_A \\ \frac{d\lambda_I}{dt} = -\frac{\partial H}{\partial I} = -1 + \beta S\lambda_S + \big(-\beta S + u_1(t)A + u_2(t)\big)\lambda_I - u_2(t)\lambda_R - u_1(t)\lambda_A \\ \frac{d\lambda_R}{dt} = -\frac{\partial H}{\partial R} = \sigma(\lambda_R - \lambda_S) \\ \frac{d\lambda_A}{dt} = -\frac{\partial H}{\partial A} = u_3(t)S\lambda_S + u_1(t)I\lambda_I - (u_3(t)S + u_1(t)I)\lambda_A \end{cases} \quad (7)$$

with transversality conditions

$$\lambda_S(T) = 0, \lambda_I(T) = 0, \lambda_R(T) = 0, \lambda_A(T) = 0 \quad (8)$$

hold.

Using the optimality condition, $\frac{\partial H}{\partial u_i(t)} = 0, i = 1,2,3$ and considering the property of the control set, the optimal control problem of equations (1)-(5) with fixed final time T_f admits a unique optimal solution $(S^*(\cdot), I^*(\cdot), R^*(\cdot), A^*(\cdot))$ associated with the optimal control $u^*(\cdot)$ on $[0, T]$ described by

$$u_1{}^*(t) = \max\left\{min\left\{\frac{1}{2C_1}A^*I^*(\lambda_I - \lambda_A), \Lambda\right\}, 0\right\} = \begin{cases} 0, u_1{}^*(t) = 0 \\ \frac{1}{2C_1}A^*I^*(\lambda_I - \lambda_A), 0 < u_1{}^*(t) < \Lambda \\ \Lambda, u_1{}^*(t) = \Lambda \end{cases}$$

$$u_2{}^*(t) = \max\left\{min\left\{\frac{1}{2C_2}I^*(\lambda_I - \lambda_R), \Lambda\right\}, 0\right\} = \begin{cases} 0, u_2{}^*(t) = 0 \\ \frac{1}{2C_2}I^*(\lambda_I - \lambda_R), 0 < u_2{}^*(t) < \Lambda \\ \Lambda, u_2{}^*(t) = \Lambda \end{cases}$$

$$u_3{}^*(t) = \max\left\{min\left\{\frac{1}{2C_3}S^*A^*(\lambda_S - \lambda_A), \Lambda\right\}, 0\right\} = \begin{cases} 0, u_3{}^*(t) = 0 \\ \frac{1}{2C_3}S^*A^*(\lambda_S - \lambda_A), 0 < u_3{}^*(t) < \Lambda \\ \Lambda, u_3{}^*(t) = \Lambda \end{cases}$$

where the adjoint functions satisfy (7) subject to the transversality conditions (8).

3. Results and Discussion

Numerical solutions from model (1) were executed using MATLAB by Runge-Kutta procedure with the following parameter values and initial conditions

$$\beta = 0.6 \; [1], \sigma = 0.05 \; [6], S(0) = 6, I(0) = 2, R(0) = 1, A(0) = 1$$

We have plotted susceptible, infected, removed, and antidotal computers with and without control. Figure 2 represents the different dynamics of the four populations of computers for two aspects of control, three aspects of control, and without control. The number of susceptible computers decreases for the three scenarios of control. The number of antidotal computers increases more rapidly than when there is no control, while the number of infected computers decreases more rapidly than when there is no control. The number of antidotal and infected computers differs much

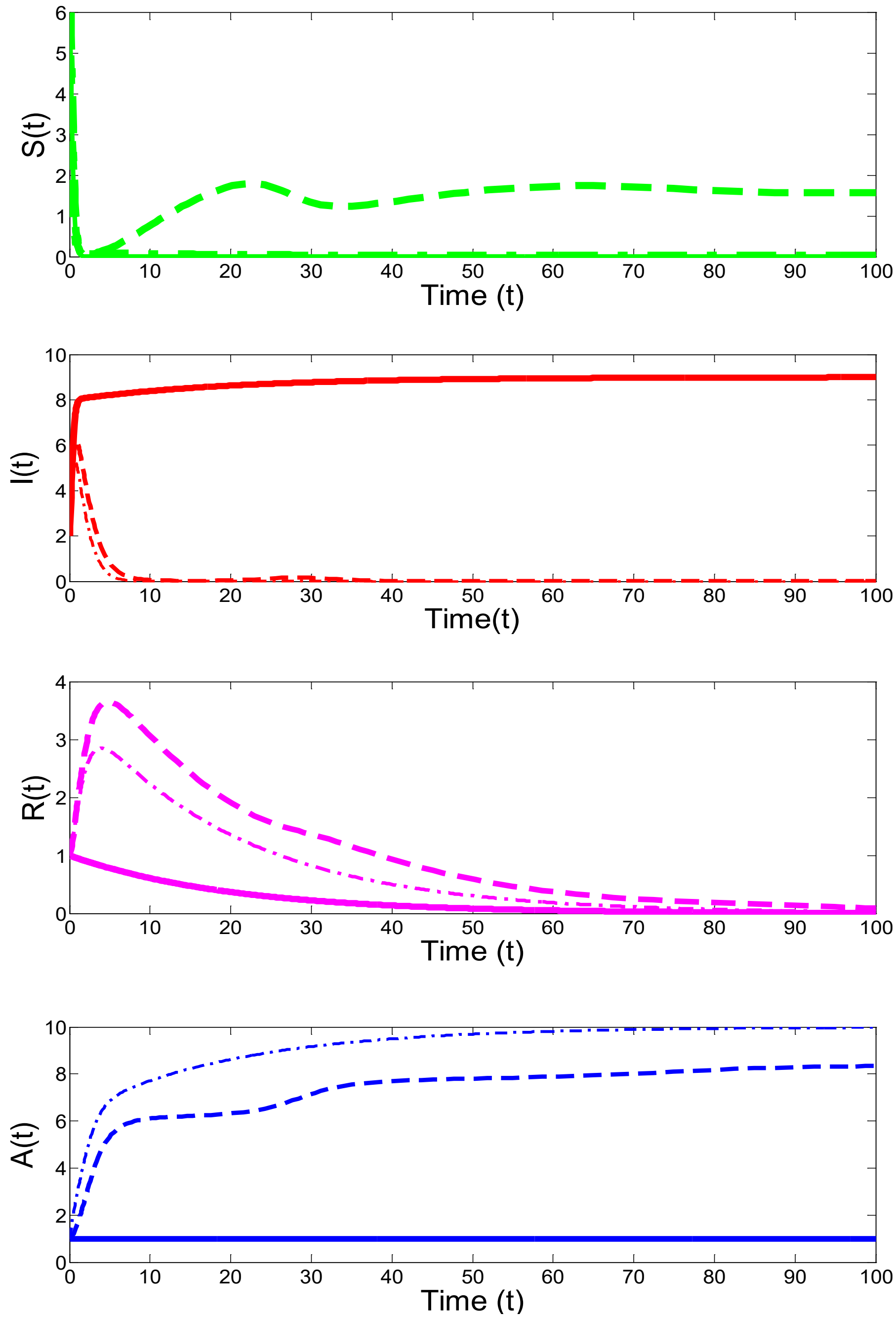

Figure 2. Optimal state variables for the control problem u_1 and u_2 (dashed line), u_1, u_2 and u_3 (dash-dotted line) versus trajectories without control measures (solid line)

from these scenarios of control if we apply the scenario of control and no control. As in susceptible computers, the number of removed computers decreases for the three scenarios of control.

The application of two controls $(u_1 \neq 0, u_2 \neq 0, u_3 = 0)$ gives better results for the number of infected computers than the application of no control while the application of three controls $(u_1 \neq 0, u_2 \neq 0, u_3 \neq 0)$ would give better results for the number of infected computers than the application of two controls. The application of three controls gives the best result for the number of infected computers. Observing the figures, the optimal control strategy for three aspects of control is more effective for the eradication of computer viruses.

Figure 3 shows the profile of the control functions u_1, u_2, and u_3 with control weight $C_i = 1, i = 1,2,3$ while Figure 4 displays numerical solutions for the model with control weight $C_i = 1, i = 1,2,3$. We observed in Figure 3, initially, we have to apply more installation of antivirus software on infected computers and susceptible computers than repaired infected computers. The results in Figure 4 show that applying more installation of antivirus software brings down the number of infected computers, which peaks at about 2.2192.

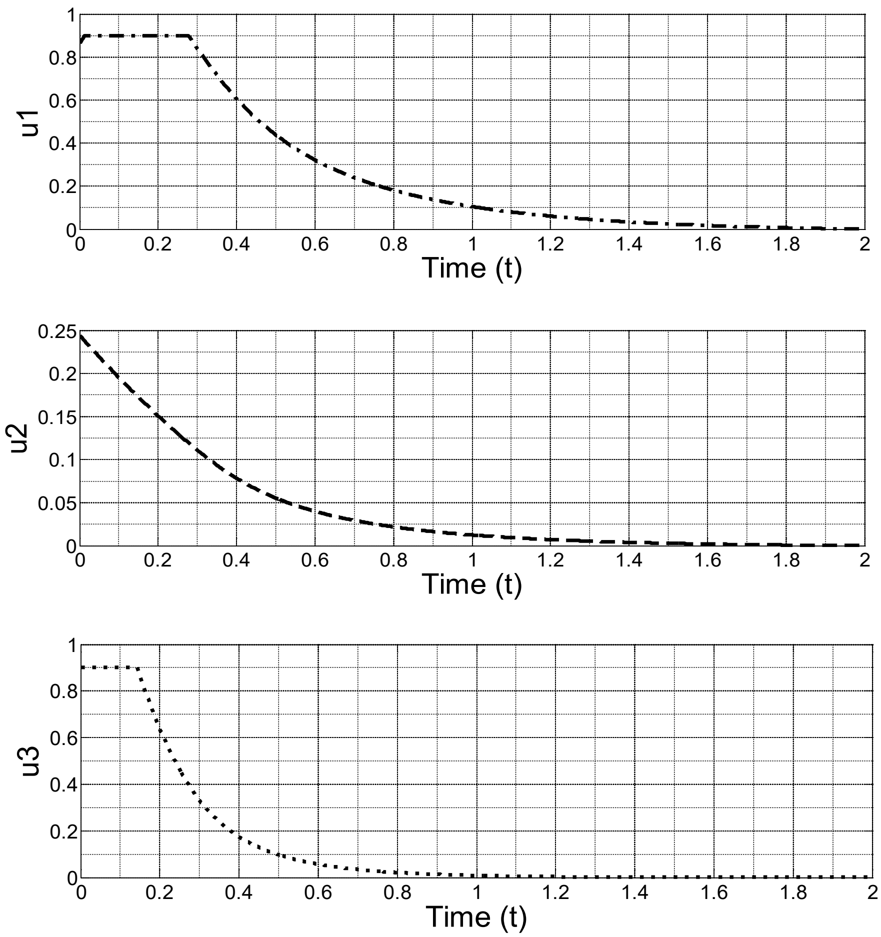

Figure 3. Optimal control u^* for the computer viruses optimal control problem (Case $C_1 = 1, C_2 = 1, C_3 = 1$)

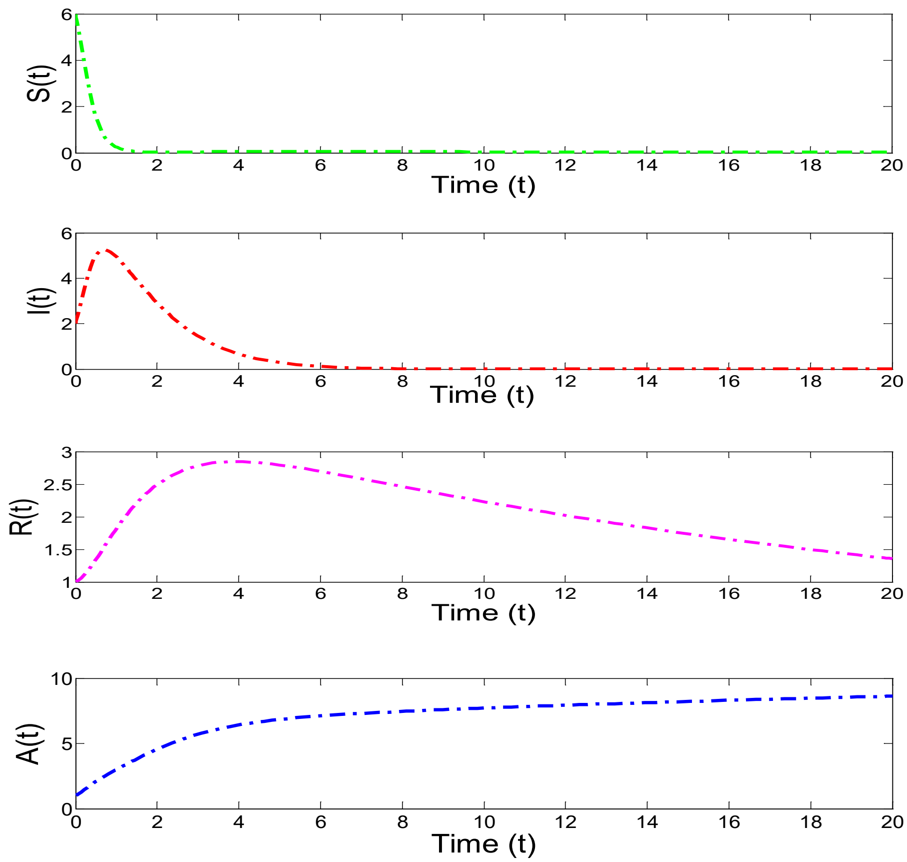

Figure 4. The dynamics of each compartment with control weight $C_1 = 1, C_2 = 1, C_3 = 1$

We used six scenarios of relative weight to observe the effects of relative weight on the application of control factors in the model. The results from the simulation of six scenarios of relative weight are displayed in Table 2.

Table 2. Optimality of control functions with six scenarios of relative weight

Scenario	C_1	C_2	C_3	$u_{1(max)}$	$u_{2(max)}$	$u_{3(max)}$	$I_{(max)}$
1	1	1	1	0.9	0.2441	0.9	2.2192
2	1	1	4	0.9	0.2223	0.5708	2.2900
3	4	1	1	0.4872	0.4258	0.8999	2.4179
4	1	4	4	0.9	0.0555	0.5755	2.3330
5	4	4	1	0.5097	0.1060	0.8999	2.5044
6	4	4	4	0.5459	0.0942	0.7941	2.5461

In Table 2, increasing the relative weight on a control factor resulted in the reduced application of rate control on the control factor because it became more expensive to implement the control.

4. Conclusions

In this paper, we investigated a model for computer virus transmission in a network system with three control factors. To maintain the number of infected computers at an optimal level, a computer virus model of deterministic type that incorporated a proportion control factor of susceptible into antidotal was formulated. We discussed here the optimal control problem for computer virus transmission, derived the conditions through the Hamiltonian, and using Pontryagin's maximum principle to achieve our main goal. As a result, the number of computers which had active viruses diminish, showed the effectiveness of our solution to the optimal control problem. Numerical solutions for different possible combinations of controls showed that an acceleration in the declining number of infected computers was achieved by giving control factors on susceptible and infected computers. Simulation results indicated that the proposed control factor was effective in reducing the number of infected computers. Moreover, an increase in relative weights will result in fewer control factors and a decrease in relative weights will result in more control factors.

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