การประมาณค่าพารามิเตอร์ของการแจกแจงทวินามลบผสมกับการแจกแจงลินด์เลย์ที่มีสองพารามิเตอร์โดยใช้วิธีภาวะน่าจะเป็นสูงสุด

Parameter Estimation of the Mixed Negative Binomial and Two-Parameter Lindley Distribution: Maximum Likelihood Method

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บทคัดย่อ

การแจกแจงทวินามลบผสมกับการแจกแจงลินดเลย์ที่มีสองพารามิเตอร์ถูกพัฒนาขึ้นเป็นการแจกแจงใหม่ที่น่าสนใจ ในบทความนี้ต้องการพัฒนาการประมาณค่าพารามิเตอร์ของการแจกแจงทวินามลบผสมกับการแจกแจงลินดเลย์ที่มีสองพารามิเตอร์โดยใช้วิธีภาวะน่าจะเป็นสูงสุด และแสดงการเปรียบเทียบประสิทธิภาพของการประมาณค่าพารามิเตอร์ดังกล่าวด้วยการจำลอง ผลการเปรียบเทียบ พบว่า การประมาณค่าด้วยวิธีภาวะน่าจะเป็นสูงสุดจะมีประสิทธิภาพสูงขึ้นเมื่อตัวอย่างมีขนาดใหญ่โดยใช้ค่าความคลาดเคลื่อนกำลังสองเฉลี่ยเป็นเกณฑ์ในการเปรียบเทียบ นอกจากนี้ได้แสดงการประยุกต์การแจกแจงทวินามลบผสมกับการแจกแจงลินดเลย์ที่มีสองพารามิเตอร์กับชุดข้อมูลจริง พบว่า การแจกแจงทวินามลบผสมกับการแจกแจงลินดเลย์ที่มีสองพารามิเตอร์มีภาวะสารูปดีกับข้อมูลได้ดีกว่าการแจกแจงปัวซง การแจกแจงทวินามลบ การแจกแจงทวินามลบผสมกับการแจกแจงลินดเลย์ เมื่อใช้ค่าพีของสถิติทดสอบแอนเดอร์สันดาร์ลิงสำหรับข้อมูลไม่ต่อเนื่องเป็นเกณฑ์ในการเปรียบเทียบ

คำสำคัญ: ภาวะน่าจะเป็นสูงสุด การประมาณค่าพารามิเตอร์ การแจกแจงทวินามลบผสมกับการแจกแจงลินดเลย์ที่มีสองพารามิเตอร์ การจำลอง สถิติทดสอบแอนเดอร์สันดาร์ลิงสำหรับข้อมูลไม่ต่อเนื่อง

Abstract

Recently, the negative binomial two-parameter Lindley distribution. In this article, the parameter estimation of the negative binomial two-parameter Lindley distribution using the maximum likelihood estimation method will be developed. A Monte Carlo simulation is applied in order to compare the efficiency of model parameter estimation using the the maximum likelihood estimation method based on mean square error of estimates. The finding results of simulation study was show that the the maximum likelihood estimates seem to have high-efficiency when the sample size becomes large. In addition, the negative binomial two-parameter Lindley distribution is applied to real data sets, we found that it can be fitted to the selected data sets. Each data set is fitted with the negative binomial two-parameter Lindley distribution, Poisson, negative binomial, and negative binomial Lindley distribution using the maximum likelihood estimation method. By comparing these results based on the p-value of Anderson-Darling goodness of fit test for fit test, it shows that the negative binomial two-parameter Lindley distribution provided the best fit when comparing to the other distributions.

Keywords: Maximum likelihood estimation, Parameter estimation, Negative binomial two-parameter Lindley distribution, Simulation study, Anderson-Darling goodness of fit test for fit test

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Introduction

Modeling of count data are found in many fields such as public health, medicine, epidemiology, applied science, sociology and agriculture. Several distributions have been proposed for this count data, especially the count data with over-dispersion. Poisson distribution was proposed by Greenwood & Yule (1920) to provide the analysis of these count data which is assumed to have the assumption of equality of mean and variance, call equi-dispersion (Cameron & Trivedi, 1998). However, many real count data show either over-dispersion, which the variance is greater than the mean or under-dispersion, which the variance is smaller than the mean (Rainer, 2000). Therefore, the negative binomial (NB) distribution originated from a mixture of Poisson and gamma distributions (Klugman, Panjer & Willmot, 2008; Lemaire, 1979) was introduced and shown more flexible alternative to fit models for count data when phenomenon of over-dispersion is presented. For such datasets, the Poisson and NB distribution cannot be used efficiently where count data with extra zeros leads to heavy tail (Wang, 2011). The Poisson distribution tends to under-estimate the number of zeros given the mean of the data, while the NB distribution may over-estimate zeros, but under-estimate observations with a count data (Lord & Geedipally, 2011). The mixing of probability distribution was introduced, in some case, it is a mixed of Poisson or mixed NB distributions, which are more flexible alternatives to analyze count data. The mixed NB distribution has been proposed by several papers, in 1981, Panjer and Willmot investigated the NB-exponential distribution by mixing NB with exponential distribution and show more flexible for fitting count data. The NB-inverse Gaussian (NB-IG) distribution was introduced (G'omez-D'eniz, Sarabia & Calderin-Ojeda, 2008), where this distribution has a thick tails, is unimodal. In 2010, Zamani and Ismail proposed the NB-Lindley (NB-L) distribution for count data, which has a thick tail and a large of zero. The NB-beta exponential distribution was proposed (Pudprommarat, Bodhisuwan & Zeephongsekul, 2012). It was obtained by mixing the NB distribution with a beta exponential distribution (Nadarajah & Kotz, 2006). In 2013, Aryuyuen and Bodhisuwan proposed the NB-generalized exponential distribution for the count data with over-dispersion and heavy tail. The NB-crack distribution (Saengthong & Bodhisuwan, 2013) was suggested for over-dispersed count data. The NB-Erlang distribution (Kongrod, Bodhisuwan & Payakkapong, 2014) was studied for the data exhibit overdispersion. Interesting, in 2016 Denthet et al., investigated a new mixed NB distribution by combining the NB and two parameter Lindley distribution (Shanker & Mishra, 2013). The NB-Sushila (NB-S) distribution was studied by Yamrubboon, et al., 2017, etc. These distributions are obtained from mixing between NB distribution and a lifetime distribution, when these distributions are used to apply for real data set. The results illustrated that these mixed NB-distribution produced better fit when compared to the Poisson and the negative binomial distributions for data with over dispersion.

In this article, the parameter estimation of NB-TPL distribution is discussed. The MLE method is applied in this study. Simulation study is conducted to illustrate of the performance of MLE based on sample sizes and some specified parameter values. In addition, the real data sets has been analyzed and compared the results with the Poisson, NB and NB-L. Finally, conclusion and discussion are provided.

Research Methodology

In this study, we want to study estimating parameter of negative binomial two-parameter Lindley distribution for simulation study and application study. Some mathematical properties of the proposed distribution is presented. Moreover, parameter estimation based on maximum likelihood is discussed.

Negative Binomial distribution

In real life trials, if each trial has two potential outcomes called **success** and **failure**, it is called Bernoulli trial. In each trial the probability of success is $0 < p < 1$ The sequence of Bernoulli trials is observed until a predefined number r of failures has occurred, where the number of successes, X is defined as the negative binomial (NB) distribution. Its probability mass function (pmf) of NB distribution is given by

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, x = 0,1,2,..., \tag{1}$$

where $r > 0$ and $0 < p < 1$ Its mean and variance are respectively,

$$E(X) = \frac{r(1-p)}{p} \quad \text{and} \quad V(X) = \frac{r(1-p)}{p^2} \tag{2}$$

Two-Parameter Lindley distribution

Let X be a random variable with TPL distribution with parameters a and b, its probability density function (pdf) (Shanker, 2013) is

$$g(x;a,b) = \frac{b^2}{b+a}(1-ax)e^{-bx}; x > 0, b > 0, a+b > 0 \tag{3}$$

Its pdf can be shown as a mixture of $g_1(x) = be^{-bx}$, exponential(b), and $g_2(x) = b^2 e^{-bx}$, gamma $(2,b)$ distributions. Consequently, it can be expressed as

$$g(x;a,b) = \theta g_1(x) + (1-\theta) g_2(x), \quad \text{where} \quad \theta = \frac{b}{b+a}$$

Moreover, the cumulative distribution function (cdf) of the TPL distribution is

$$G(x) = 1 - \frac{b+a+abx}{b+a} e^{-bx}; x > 0, b > 0, a+b > 0$$

Result

In this study, the negative binomial two-parameter Lindley distribution is proposed, the theorem and detailed are as follow.

Theorem 1 If $X \sim \text{TPL}(a,b)$ then the moment generating function (mgf) of the TPL distribution is

$$M_x(t) = \frac{(a+b-t)b^2}{(b+a)(b-t)^2}, \quad \text{where } a+b > 0 \text{ and } b > 0 \tag{4}$$

Proof. Let $X \sim \text{TPL}(a,b)$, then the mgf of TPL distribution can be obtain by

$$M_x(t) = \int_0^\infty e^{tx} g_x(x) dx \quad = \quad \int_0^\infty e^{tx} \left(\frac{b^2}{b+a}\right)(1+ax)e^{-\theta x} dx$$

$$= \frac{b^2}{b+a}\left[\int_0^\infty e^{-(\theta-t)x} dx + a\int_0^\infty xe^{-(\theta-t)x} dx\right].$$

$$= \frac{b^2}{b+a}\left[\frac{1}{b-t} + \frac{a}{(b-t)^2}\right],$$

Then $M_x(t)$ can be expressed as

$$M_x(t) = \frac{(a+b-t)b^2}{(b+a)(b-t)^2}.$$

Some plots of TPL pdf are show in Figure 1. We can see that if a is small, then the probability of occurrence close to 0. When a becomes larger the pdf plot is similar to a right skewed distribution. When b is increasing, the tails close to 0 on the right.

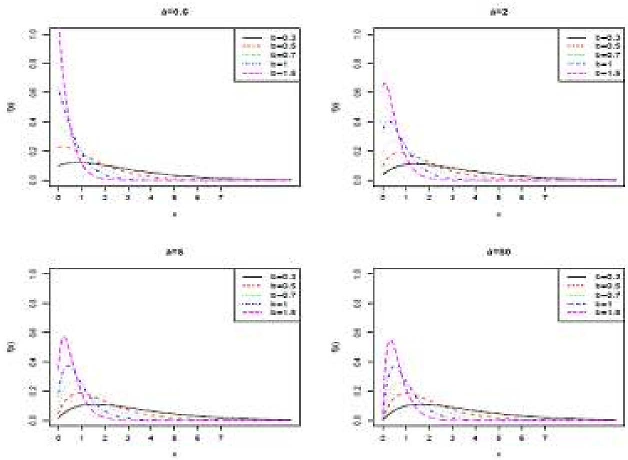

Figure 1: shows some plots of the TPL pdf with specified parameters ***a*** and ***b***.

A New Mixed NB distribution

We propose a new mixed NB distribution between the NB distribution with the parameters r and p. The TPL distribution with the positive parameters a and $b,$ namely, NB-TPL distribution.

Definition 1 Let X be a random variable of the NB distribution with the parameters r and $p = \exp(-\lambda),$ where λ is distributed as the TPL distribution with the positive parameters a and $b,$ i.e., $X \mid \lambda \sim \mathrm{NB}(\mathrm{r}, \mathrm{p} = \exp(-\lambda))$ and $\lambda \sim \mathrm{TPL}(a,b)$.

Theorem 2 Let X be a random variable of the NB-TPL distribution with parameters a and $b,$ will be denoted as $X \sim \mathrm{NB\text{-}TPL}(r,a,b)$. The pmf of X is

$$f(x;a,b) = \binom{r+x-1}{x} \sum_{j=0}^{x} \binom{x}{j} (-1)^j \frac{(a+b+r+j)b^2}{(b+a)(b+r+j)^2}, x = 0,1,2,... \quad (5)$$

where $a+b>0$ and $b>0$

Proof. Let X be NB distribution with the parameters r and p with pmf.

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, x = 0,1,2,...$$

and $X \mid \lambda \sim \mathrm{NB}(r, p = \exp(-\lambda))$ where $\lambda \sim \mathrm{TPL}(a,b)$ we obtain the pmf of the NB-TPL distribution as

$$f(x;r,a,b) = \int_0^{\infty} f(x;\lambda) \cdot g(\lambda;a,b)\, d\lambda, \quad (6)$$

where $$f(x;\lambda) = \binom{r+x-1}{x} e^{-r\lambda}(1-e^{-\lambda})^x \quad (7)$$

Substitution of $f(x;\lambda)$ in (7) into (6),

$$f(x;r,a,b)=\int_0^\infty \binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j e^{-\lambda(r+j)}\cdot g(\lambda;a,b)d\lambda,$$

$$=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j\int_0^\infty e^{-\lambda(r+j)}\cdot g(\lambda;a,b)d\lambda,$$

$$=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j M_\lambda(-(r+j)).$$

From the mgf of TPL distribution in (2) we obtain the pmf of X as

$$f(x;r,a,b)=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j\frac{(a+b+r+j)b^2}{(b+a)(b+r+j)^2}$$

Figure 2 shows some plots of pmf of the NB-TPL distribution with some specified parameters r,a and b We illustrate the possible shapes of NB-TPL pmf. When r is small, the probability of occurrence will close to 0. When r becomes higher, the center of the curve moves toward the right. When b is increasing, the tails become heavier on the right.

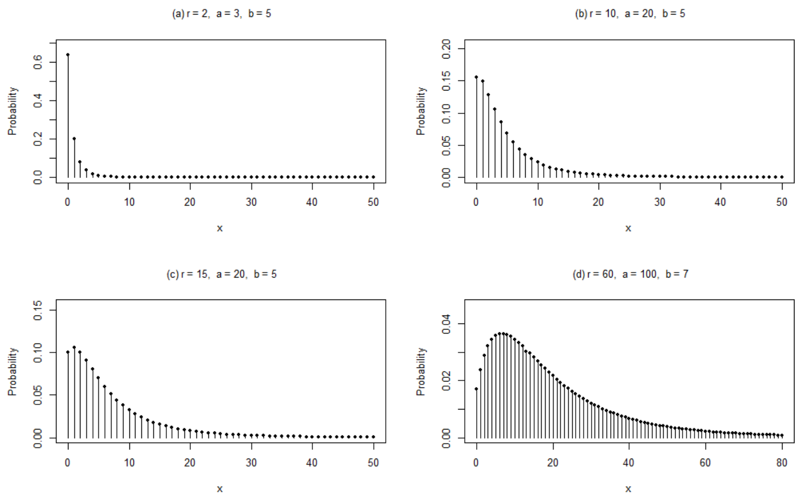

Figure 2: Some plots of the NB-TPL distribution.

The special cases of the NB-TPL distribution will be presented in the following corollaries.

Corollary 1 If $a=1$ then the NB-TPL distribution reduces to the NB-L distribution which proposed by Zamani & Ismail (2010), with pmf given by

$$f(x;r,b)=\frac{b^2}{b+1}\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j\frac{(b+r+j+1)}{(b+r+j)^2},\ x=0,1,2,\ldots,\ \text{and}\ r,b>0. \quad (8)$$

Corollary 2 If $a=0$ then the NB-TPL distribution reduces to the NB-E distribution which proposed by Panjer and Willmot (1981), with pmf given by

$$f(x;r,b)=\binom{r+x-1}{x}\sum_{j=0}^{x}\binom{x}{j}(-1)^j\frac{b}{(b+r+j)},\ x=0,1,2,\ldots,\ \text{and}\ r,b>0. \quad (9)$$

Corollary 3 If $r=1, a=0$ then the NB-TPL distribution reduces to the geometric–exponential distribution which proposed by Panjer & Willmost (1981), with pmf given by

$$f(x;b)=\sum_{j=0}^{x}\binom{x}{j}(-1)^j\frac{b}{(b+j+1)},\ x=0,1,2,\ldots,\text{ and } b>0. \tag{10}$$

Random variate generation of NB-TPL distribution

To generate a random variate X of NB-TPL can use the following algorithm.

1) Generate u from the uniform distribution $\mathrm{U}(0,1)$

2) Generate λ from $\mathrm{TPL}(a,b)$

a. Set $\theta=\dfrac{b}{b+a}$

b. If $u\le\theta$ then λ generate from the exponential (b) distribution.

c. If $u>\theta$ then λ generate from the gamma $(2,b)$ distribution.

3) Generate X from the $\mathrm{NB}(r, p=\exp(-\lambda))$ distribution.

Parameter estimation of NB-TPL distribution

In this section, the parameter estimation of NB-TPL distribution via MLE procedure is provided. The likelihood function of $\mathrm{NB\text{-}TPL}(r,a,b)$ is given by

$$L(r,a,b)=\prod_{i=1}^{n}\left[\frac{b^2}{b+a}\binom{r+x_i-1}{x_i}\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{b+a+r+j}{(b+r+j)^2}\right],$$

then we can write the log-likelihood of the $\mathrm{NB\text{-}TPL}(r,a,b)$, $\pounds=\log L(r,a,b)$ as

$$\pounds=\sum_{i=1}^{n}\log\left[\frac{b^2}{b+a}\frac{\Gamma(r+x_i)}{\Gamma(r)\Gamma(x_i+1)}\sum_{j=0}^{x_i}\binom{x_i}{j}(-1)^j\frac{b+a+r+j}{(b+r+j)^2}\right],$$

$$=n\log b^2-n\log(b+a)-n\log\Gamma(r)+\sum_{i=1}^{n}\log\Gamma(r+x_i)-\sum_{i=1}^{n}\log\Gamma(x_i+1)$$

$$+\sum_{j=0}^{x_i}\log\binom{x_i}{j}(-1)^j\frac{b+a+r+j}{(b+r+j)^2}$$

By differentiating the log-likelihood function of NB-TPL distribution, partial derivatives of the log likelihood function with respect to r, a and b are given by

$$\frac{\partial\pounds}{\partial r}=-n\psi(r)+\sum_{i=1}^{n}\psi\Gamma(r+x_i)+\sum_{j=0}^{x_i}\frac{\partial}{\partial r}\log\binom{x_i}{j}(-1)^j\frac{b+a+r+j}{(b+r+j)^2},$$

$$\frac{\partial\pounds}{\partial a}=-n\left(\frac{1}{b+a}\right)+\sum_{j=0}^{x_i}\frac{\partial}{\partial a}\log\binom{x_i}{j}(-1)^j\frac{b+a+r+j}{(b+r+j)^2}$$

$$\frac{\partial\pounds}{\partial b}=-n\left(\frac{2a+b}{b(b+a)}\right)+\sum_{j=0}^{x_i}\frac{\partial}{\partial b}\log\binom{x_i}{j}(-1)^j\frac{b+a+r+j}{(b+r+j)^2}$$

where $\psi(s)=\Gamma'(s)/\Gamma(s)$ is a digamma function.

The MLE estimator of the parameters of the proposed distribution can be obtained by using numerical optimization with the ***nlm*** function in the R program (R Core Team, 2015).

Simulation study

This section presents the efficiency of the MLE method for parameter estimation of NB-TPL distribution by using simulated data. In addition, we illustrate the application study of the NB-TPL distribution compared to the Poisson, NB and NB-L distributions.

Table 1 All cases of simulation study.

Case	Parameter	Mean	S.D.
1	$r = 2, a = 3, b = 5$	0.73	1.51
2	$r = 10, a = 20, b = 5$	5.00	6.41
3	$r = 15, a = 20, b = 5$	7.50	9.20
4	$r = 60, a = 100, b = 7$	20.90	20.33

To illustrate the simulation study, the sample data generated from the NB-TPL distribution with specified parameters in four cases, shown in Table 1, are used. In each situation, the sample sizes (n) as 20, 50, 100 and 200 are used. The parameters are estimated from 500 replications. The sample average of the estimated parameter, bias, variance and mean squared error (MSE) are computed by these formulas, respectively:

$$\hat{\theta}_{av} = \frac{1}{500}\sum_{t=1}^{500}\hat{\theta}_t \text{ , Bias}(\hat{\theta}_{av}) = \hat{\theta}_{av} - \theta \text{ , Var}(\hat{\theta}_{av}) = \frac{1}{500-1}\sum_{t=1}^{500}(\hat{\theta}_t - \hat{\theta}_{av})^2,$$

$$\text{SD}(\hat{\theta}_{av}) = \sqrt{\text{Var}(\hat{\theta}_{av})} \text{ and } \text{MSE}(\hat{\theta}_{av}) = \text{Var}(\hat{\theta}_{av}) + \text{Bias}^2(\hat{\theta}_{av})$$

Table 2 illustrate statistic values from the results of studies. In Figure 3, MLE seems to have high-efficiency when the sample size is large. Hence the efficiency of this method seems to be poor for small sample sizes.

Table 2 The parameter estimate of the NB-TPL distribution.

Parameter	n=20		n=50		n=100		n=200	
	Estimate	MSE	Estimate	MSE	Estimate	MSE	Estimate	MSE
r=2	3.16	15.76	2.96	12.7	2.56	3.91	2.42	1.67
a=3	4.07	26.91	3.71	24.49	3.18	10.59	3.03	7.64
b=5	5.84	21.65	5.74	12.75	5.53	8.12	5.28	2.57
r=10	13.36	58.55	11.84	26.27	11.43	18.41	10.55	5.24
a=20	21.5	40.34	21.63	38.85	21.31	25.69	19.99	9.21
b=5	6.21	5.5	6.07	4.73	5.61	3.94	5.54	2.57
r=15	16.70	62.52	16.14	47.14	15.6	40.35	15.1	29.33
a=20	22.14	14.93	21.27	11.02	20.95	8.20	20.03	1.36
b=5	6.14	6.11	5.99	4.18	5.84	2.73	5.00	0.71
r=60	64.05	67.33	63.39	60.45	62.04	37.47	59.77	33.77
a=100	104.33	35.18	103.13	25.01	101.95	20.28	100.96	16.60
b=7	8.55	5.70	8.01	3.34	7.60	2.52	7.06	1.13

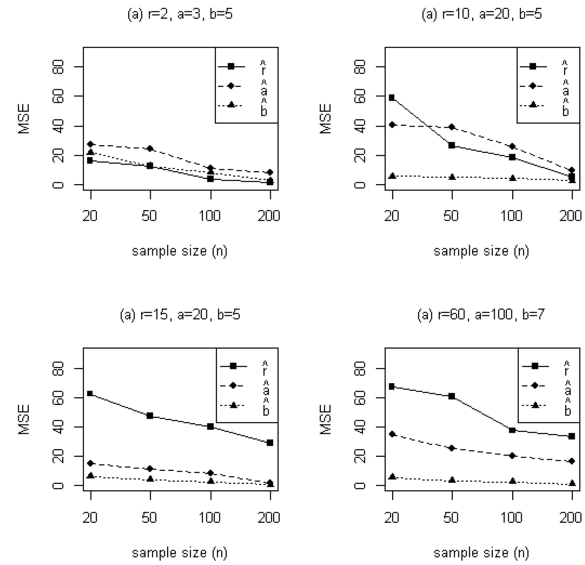

Figure 3: The MSE of parameter estimates of NB-TPL distribution.

Application

We used the datasets for this part of the analysis to illustrate the applications of the NB-TPL distribution. The first dataset is single-vehicle fatal crashes that occurred on divided multilane rural highways between 1997 and 2001 (Lord & Geedipally, 2011). The second dataset is the number of single-vehicle roadway departure fatal crashes that occurred on 32,672 rural two□lane horizontal curves between 2003 and 2008 (Lord & Geedipally, 2011). The datasets are presented in Table 3-4 with the number of observed frequencies and expected value by fitting distribution.

For model selection in this study, we use the criteria such as AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), and Anderson–Darling (AD) goodness-of-fit test for discrete data. test. From the results in Table 3-4, we found that AD values of the NB-TPL distribution are smallest when compared with the existing models. Also, based on the p-values of AD test, the proposed distribution is appropriate to fit the datasets. Thus, the NB-TPL distribution can be chosen as the best model.

Table 3 Fitting distribution of single-vehicle fatal crashes data set.

Number of Crashes	Observed frequency	Expected value by fitting distribution			
		Poisson	NB	NB-L	NB-TPL
0	1532	1499.2	1550.6	1530.3	1533.1
1	162	206.9	113.5	160.6	158.6
2	19	14.3	36.1	24.4	23.8
3	6	0.7	14.4	4.7	4.5
4	2	0.0	6.4	1.1	1.0
Total	**1721**				
Parameters		$\hat{\lambda}$ =0.131	$\hat{r}$ =0.474 $\hat{p}$ =0.783	$\hat{r}$ =1.881 $\hat{b}$ =16.211	$\hat{r}$ =1.896 $\hat{a}$ =0.001 $\hat{b}$ =15.441
AIC		1432	1396	1395	1397
BIC		1437	1406	1406	1413
AD-test		2.607	0.037	0.021	0.020
P-value		0.024	0.914	0.955	0.956

In Table 3, the real data set which is single-vehicle fatal crashes (Lord & Geedipally, 2011) has been fitted by Poisson, NB, NB-L and NB-TPL distributions. The MLE method provided parameter estimates of these distributions. By comparing these results based on the p-value of the Anderson-Darling goodness-of-fit test for discrete data also AIC and BIC, it showed that the NB-TPL distribution provided the best fit among the other distributions.

Table 4 Fitting distribution of single-vehicle roadway departure fatal crashes data set.

Number of Crashes	Observed frequency	Expected value by fitting distribution			
		Poisson	NB	NB-L	NB-TPL
0	29087	28489.0	27495.1	24261.6	29117.2
1	2952	3903.0	2716.7	2845.9	2887.7
2	464	267.4	1106.8	1472.7	501.0
3	108	12.2	564.8	980.1	116.3
4	40	0.4	317.2	728.4	32.7
5	9	0.0	188.0	576.4	10.6
6	5	0.0	115.3	475.0	3.8
7	2	0.0	72.4	402.7	1.5
8	3	0.0	46.2	348.7	0.6
9	1	0.0	29.9	306.9	0.3
10	1	0.0	19.6	273.6	0.1
Total	**32672**				

Table 4 Fitting distribution of single-vehicle roadway departure fatal crashes data set.

Number of Crashes	Observed frequency	Expected value by fitting distribution			
		Poisson	NB	NB-L	NB-TPL
Parameters		$\hat{\lambda}$ =0.138	$\hat{r}$ =0.343 $\hat{p}$ =0.714	$\hat{r}$ =1.131 $\hat{b}$ =10.055	$\hat{r}$ =1.147 $\hat{a}$ =0.0012 $\hat{b}$ =9.344
AIC BIC		28418 28426	27103 27120	27061 27078	23061 23062
AD-test P-value		89.695 <0.001	0.013 0.617	0.128 0.713	0.126 0.717

In Table 4, an another real data set has been fitted by Poisson, NB, NB-L and NB-TPL distributions using the MLE method. The parameter estimates of these distributions are shown in the table. Based on the p-value of the Anderson-Darling goodness-of-fit test for discrete data also AIC and BIC, it showed that the NB-TPL distribution provided the best fit among the other distributions.

Conclusion

The NB-TPL distribution, is obtained by mixing the NB distribution with survival model of TPL distribution, of which the NB-L, NB-E and geometric-exponential distribution are particular case. The closed form and some characteristics of the proposed distribution are studied. A parameter estimation is also implemented by using MLE, and the usefulness of the NB-TPL distribution is illustrated by real and generated data. Also, the MLE method seems to have high-efficiency for datasets characterized by a large number of zeros and a heavy tail. Based on the results of the application, the proposed distribution can be an alternative for count ata analysis, it is conceived that the NB-TPL distribution may become a really useful tool for analyzing count data characterized with a large number of zeros.

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