Binary systems of full terms arising from some mappings

Thodsaporn Kumduang[1*]

Abstract

Full terms with an invariant set are special types of terms defined by full transformations with an invariant set on a finite set and variables from an alphabet applied in the theory of solid varieties. The set of all full terms with an invariant set is closed under the superposition operation under which the superassociative law holds. This work introduced three different binary operations on the set of all full terms with an invariant set and proves associativity. Moreover, tree languages of full terms with an invariant set and their operations were considered. Finally, embedding theorems of semigroups of full terms with an invariant set into semigroups of tree languages of full terms with an invariant set were proposed.

Keywords: full term, tree language, associativity

[1] Faculty of Science and Technology, Rajamangala University of Technology Rattanakosin

[*] Corresponding author. E-mail: thodsaporn.kum@rmutr.ac.th

Received: April 22, 2023; Revised: August 29, 2023; Accepted: September 13, 2023

Introduction

First, we recall from (Denecke, 2016; Chansuriya, 2021; Kumduang, & Sriwongsa, 2022) that terms, sometimes known as trees, are expressions generated by variables from an alphabet and compositions of fundamental symbols. By definition, an n-ary term of type τ is inductively defined by the following steps: Each variable x_i in $X_n := \{1, \ldots, n\}$ is an n-ary term of type τ and $f_i(t_1, \ldots, t_{n_i})$ is also an n-ary term of type τ if $t_1, \ldots, t_{n_i}$ are already known. Recent studies in several classes of terms can be found, for example, in (Joomwong, & Phusanga, 2021; Lekkoksung, & Lekkoksung, 2021). Furthermore, each term has a tree representation. For instance, a tree representation of a term $f(x_7, g(h(x_1, x_2)), h(x_3, x_5))$ can be shown in the (Figure 1).

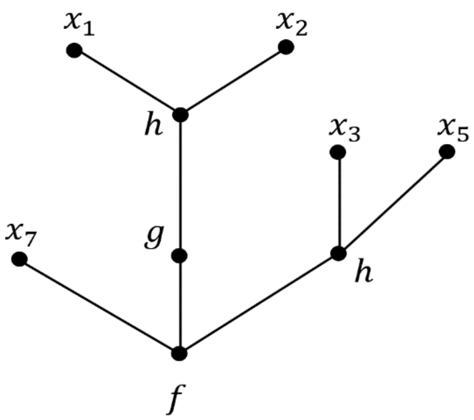

Figure 1 A tree representation of a term $f(x_7, g(h(x_1, x_2)), h(x_3, x_5))$.

One of the outstanding classes of terms is a full term introduced by K. Denecke and his colleagues in (Denecke, & Jampachon, 2003). To attain this, let τ_n be a type of the operation symbol of all arity n for all $i \in I$, i.e., $\tau_n = (n_i)$ and $n_i = n$ for all $i \in I$. By the symbol T_n, we denote the set of all transformations from $\overline{n} = \{1, \ldots, n\}$ to itself. Actually, the set T_n together with a binary composition of functions forms a semigroup called a transformation semigroup. Applying this concept, for any mapping α in T_n, an n-ary full term of type τ_n is defined by the following steps:

1. $f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$ is an n-ary full term of type τ_n where $\alpha \in T_n$,

2. if $t_1, \ldots, t_n$ are n-ary full terms of type τ_n, then $f_i(t_1, \ldots, t_n)$ is an n-ary full term of type τ_n. The set of all n-ary full terms of type τ_n is denoted by $W^F_{\tau_n}(X_n)$.

Let us consider some example. For a type $\tau_3 = (3, 3)$ with two ternary operation symbols, say f and g, we have $f(x_1, x_2, x_3)$, $f(x_1, x_1, x_1)$, $f(\ g(x_1, x_2, x_3),\ f(x_3, x_1, x_2),\ f(x_2, x_1, x_3))$ are examples of full terms in the set $W^F_{(3,3)}(X_n)$.

However, there are various ways to study terms in a higher step. One of the generalizations of terms in the study of universal algebra and automata theory is a set of terms. In fact, we call sets of terms tree languages, see (Salehia, & Steinby, 2007). The set of all subsets or tree languages of all n-ary full terms of type τ_n is denoted by $P(W_{\tau_n}^F(X_n))$. For example, we have $\{f(x_1, x_2, x_3)\}, \{g(x_1, x_1, x_1), f(x_2, x_2, x_2)\}$, $\{f\big(g(x_1, x_2, x_3), f(x_3, x_1, x_2), f(x_2, x_1, x_3)\big)\}$ are examples of tree languages in $P(W_{(3,3)}^F(X_n))$. Nevertheless, a set $\{f(x_1, f(x_1, x_2, x_3), x_3)\}$ is not a tree language of ternary full terms of type $(3,3)$. To compute the result of tree languages of full terms, in (Wattanatripop, & Changphas, 2021a), a non-deterministic superposition operation on the set $P(W_{\tau_n}^F(X_n))$ was defined. By definition, a mapping $\hat{S}^n: P\left(W_{\tau_n}^F(X_n)\right)^{n+1} \to P\left(W_{\tau_n}^F(X_n)\right)$ is defined as follows:

1. $\hat{S}^n\big(\{f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})\}, B_1, \dots, B_n\big) = \{f_i(r_{\alpha(1)}, \dots, r_{\alpha(n)}) | r_{\alpha(j)} \in B_{\alpha(j)}, j = 1, \dots, n\}$,

2. $\hat{S}^n(\{f_i(t_1, \dots, t_n)\}, B_1, \dots, B_n) = \{f_i(r_1, \dots, r_n) | r_j \in \hat{S}^n(\{t_j\}, B_1, \dots, B_n), j = 1, \dots, n\}$,

3. if $|A| > 1$, then $\hat{S}^n(A, B_1, \dots, B_n) = \bigcup_{a \in A}\{\hat{S}^n(\{a\}, B_1, \dots, B_n)\}$,

4. $\hat{S}^n(A, B_1, \dots, B_n) = \emptyset$ if $A = \emptyset$ or $B_j = \emptyset$ for some j.

Recall that a nonempty set G and an operation o defined on G satisfying the superassociative law, i.e.,

$$o(o(a, b_1, \dots, b_n), d_1, \dots, d_n) = o(a, o(b_1, d_1, \dots, d_n), \dots, o(b_n, d_1, \dots, d_n))$$

is called a Menger algebra. As a consequence, the Menger algebra $(P\left(W_{\tau_n}^F(X_n)\right), \hat{S}^n)$ of type $(n+1)$ is obtained.

In 2022, the concept of full terms with an invariant set was introduced by in (Phuapong, & Pookpienlert, 2022). We now recall the definition of a semigroup of transformations with an invariant set introduced in (Honyam, & Sanwong, 2011). For a fixed nonempty subset Y of X, the set $S(X, Y) = \{\alpha \in f: X \to X \mid Y\alpha \subseteq Y\}$ whose elements are called transformations with an invariant set equipped with the usual composition of functions is a semigroup. For more details, we refer to (Chinram, & Baupradist, 2019; Sarkar, & Singh, 2022). Applying this structure, a particular class of full terms was given. Actually, an n-ary $S(\overline{n}, Y)$-full term of type τ_n is inductively defined in the following setting:

1. $f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ is an n-ary $S(\overline{n}, Y)$-full term of type τ_n where $\alpha \in S(\overline{n}, Y)$,

2. if $t_1, \dots, t_n$ are n-ary $S(\overline{n}, Y)$-full terms of type τ_n, then $f_i(t_1, \dots, t_n)$ is an n-ary $S(\overline{n}, Y)$-full term of type τ_n.

3. The set $W_{\tau_n}^{S(\overline{n}, Y)}(X_n)$ of all n-ary $S(\overline{n}, Y)$-full terms of type τ_n is the smallest set containing $f_i(t_{\alpha(1)}, \dots, t_{\alpha(n)})$ and is closed under finite application of 2.

For example, let $\tau_3 = (3,3,3)$ be a type with three ternary operation symbols $\otimes, \boxtimes, \odot$. For a fixed subset $Y = \{1,3\}$ of $\overline{3}$, we have $\otimes(x_1, x_2, x_3), \boxtimes(x_1, x_3, x_1) \in W_{(3,3,3)}^{S(\overline{3},\{1,3\})}(X_3)$ because $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix} \in S(\overline{3}, \{1,3\})$. On the other hand, it is not difficult to see that $\otimes(x_2, x_1, x_1), \odot(x_2, x_2, x_2) \notin W_{(3,3,3)}^{S(\overline{3},\{1,3\})}(X_3)$

because $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$ do not belong to the set $S(\overline{3},\{1,3\})$.

The superposition operation S^n on the set $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ of all n-ary $S(\overline{n},Y)$-full terms of type τ_n was mentioned in the paper (Wattanatripop, & Changphas, 2019, Wattanatripop, & Changphas, 2021b). In fact, it is a mapping $S^n: W_{\tau_n}^{S(\overline{n},Y)}(X_n)^{n+1} \to W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ defined by

1. $S^n(f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)}), s_1, \dots, s_n) = f_i(s_{\alpha(1)}, \dots, s_{\alpha(n)})$,
2. $S^n(f_i(t_1, \dots, t_n), s_1, \dots, s_n) = f_i(S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))$.

As a result, the algebra $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), S^n\right)$ of type $(n+1)$ is constructed. It is not hard to verify that the operation S^n defined on $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ satisfies the following identity:

$$S^n(S^n(t, s_1, \dots, s_n), u_1, \dots, u_n) = S^n(t, S^n(s_1, u_1, \dots, u_n), \dots, S^n(s_n, u_1, \dots, u_n)).$$

This equation also known as the superassociative law which always plays a key role in the theory of multiplace functions and Menger algebras. For more details, we refer to (Denecke, & Hounnon, 2021; Dudek, & Trokhimenko, 2021; Phuapong, & Kumduang, 2021; Denecke, 2022).

In this work, based on a direction of the papers (Phuapong, & Kumduang, 2021, Wattanatripop, & Changphas, 2021b) we aim to continue the study of $S(\overline{n},Y)$-full terms by introducing binary operations on the set $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ of all $S(\overline{n},Y)$-full terms derived from the superposition S^n and prove that these operations are associative. For tree languages of $S(\overline{n},Y)$-full terms, binary operations induced by a non-deterministic operation $\hat{S}^n$ are introduced and the fact that these operations satisfy an associativity is given. Finally, this work is devoted to the embeddability of semigroups of $S(\overline{n},Y)$-full terms into semigroups of tree languages of $S(\overline{n},Y)$-full terms.

Methodology

We begin this section with giving three kinds of binary operations defined on the set of all $S(\overline{n},Y)$-full terms. For any s,t in $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$, we define the binary operation $+: W_{\tau_n}^{S(\overline{n},Y)}(X_n) \times W_{\tau_n}^{S(\overline{n},Y)}(X_n) \to W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ By $s + t = S^n(s, \underbrace{t \dots, t}_{n})$. Then we have:

Theorem 1 $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), +\right)$ is a semigroup.

Proof Let s, t, u be $S(\overline{n},Y)$-full terms of type τ_n. Because the operation S^n satisfies the superassociativity, we have $(s+t)+u = S^n(s,t,\dots,t) + u = (S^n(s,t,\dots,t), u, \dots, u) = S^n(s, S^n(t,u,\dots,u), \dots, S^n(t,u,\dots,u)) = s + S^n(t,u,\dots,u) = s + (t+u)$. As a consequence, $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), +\right)$ forms a semigroup.

The semigroup given in Theorem 1 is called the *diagonal semigroup* derived from the Menger algebra $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), S^n\right)$. For more details, see (Dudek, & Trokhimenko, 2021).

Let s and t be $S(\overline{n},Y)$-full terms in the set $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$. For each $i = 1,\dots,n$, the binary operation $\cdot_{x_i}: W_{\tau_n}^{S(\overline{n},Y)}(X_n) \times W_{\tau_n}^{S(\overline{n},Y)}(X_n) \to W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ can be defined by $s \cdot_{x_i} t = S^n(s, x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)$.

We now prove the fact that the binary operation $\cdot_{x_i}$ satisfies the associative law.

Theorem 2 $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), \cdot_{x_i}\right)$ is a semigroup.

Proof To prove that the binary operation $\cdot_{x_i}$ is associative, let s,t,u be arbitrary full terms in $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$. Due to the satisfaction of S^n of the superassociative law over the set $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$, we obtain $(s \cdot_{x_i} t) \cdot_{x_i} u = s \cdot_{x_i} (t \cdot_{x_i} u)$, which shows that $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), \cdot_{x_i}\right)$ forms a semigroup.

For a positive integer n, on the Cartesian product $W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n$ of n-tuples of $S(\overline{n},Y)$-full terms of type τ_n, the binary operation $*$: $W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n \times W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n \to W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n$ can be defined by $(s_1,\dots,s_n) * (t_1,\dots,t_n) = (S^n(s_1,t_1,\dots,t_n),\dots,S^n(s_n,t_1,\dots,t_n))$ for all $(s_1,\dots,s_n),(t_1,\dots,t_n) \in W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n$.

As a consequence, we prove the following theorem.

Theorem 3 $(W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n, *)$ is a semigroup.

Proof It follows from a direct verification.

On the set $W_{\tau_n}^{S(\overline{n},Y)}(X_n)$, one can consider its power set, i.e., $P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$. Each element in $P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)$ is called a *tree language of* $S(\overline{n},Y)$ *-full terms.* For example, consider a type $\tau_4 = (4)$ with one quaternary operation symbol $\boxplus$ and a set $Y = \{1,4\} \subseteq \overline{4}$. It is clear that $\emptyset, \{\boxplus(x_1,x_2,x_3,x_4)\}, \{\boxplus(x_4,x_4,x_4,x_4)\}, \{\boxplus(x_4,x_3,x_1,x_1), \boxplus(x_1,x_4,x_2,x_1)\}$ are examples of tree languages in $P(W_{\tau_4}^{S(\overline{4},\{1,4\})}(X_4))$.

Then we prove the following result.

Theorem 4 $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right), \hat{S}^n\right)$ is a subalgebra of the power Menger algebra $\left(P\left(W_{\tau_n}^{F}(X_n)\right), \hat{S}^n\right)$.

Proof It is easy to see that $P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)$ is a subset of $P\left(W_{\tau_n}^{F}(X_n)\right)$. Now we let $A, B_1, \dots, B_n$ be subsets of $S(\overline{n},Y)$-full terms. We give a proof by a structure of a set A. If one of the sets $A, B_1, \dots, B_n$ is an empty set, then $\hat{S}^n(A, B_1, \dots, B_n) = \emptyset$, which implies that $\hat{S}^n(A, B_1, \dots, B_n) \in P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)$. If A is a one element set, we consider in two cases. If $A = \{f_i(x_{\alpha(1)},\dots,x_{\alpha(n)})\}$ where α is a mapping on $S(\overline{n},Y)$, then we have that $\hat{S}^n(\{f_i(x_{\alpha(1)},\dots,x_{\alpha(n)})\}, B_1,\dots,B_n)$ which equals to $\{f_i(r_{\alpha(1)},\dots,r_{\alpha(n)}) \mid r_{\alpha(j)} \in B_{\alpha(j)}, j = 1,\dots,n\}$ belongs to the set $P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$ because $B_{\alpha(j)} \in P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$ for every $j = 1,\dots,n$. Suppose now that $A = \{f_i(t_1,\dots,t_n)\}$ and that each $\hat{S}^n(\{t_j\}, B_1,\dots,B_n)$ is an $S(\overline{n},Y)$-full term of type τ_n on $S(\overline{n},Y)$. It follows that $\hat{S}^n(\{f_i(t_1,\dots,t_n)\}, B_1,\dots,B_n) = \{f_i(r_1,\dots,r_n) \mid r_j \in \hat{S}^n(\{t_j\}, B_1,\dots,B_n), j = 1,\dots,n\}$ is in the set $P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$. In the case when a set A is a nonempty arbitrary set, we obtain that

$\hat{S}^n(A, B_1, \dots, B_n) = \bigcup_{a\in A}\{\hat{S}^n(\{a\}, B_1, \dots, B_n)\}$ is in $P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$ because each $\hat{S}^n(\{a\}, B_1, \dots, B_n)$ is already known. Consequently, the proof is finished.

According to Theorem 4, we can remark here that the operation $\hat{S}^n$ satisfies a superassociative law over the set $P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$.

Let A and B be two elements of $P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$. Then we define the binary operation $\hat{+}$ on $P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$ by $A\hat{+}B = \hat{S}^n(A, B_1, \dots, B_n)$. The following theorem shows that $\hat{+}$ is associative.

Theorem 5 $\left(P(W_{\tau_n}^{S(\overline{n},Y)}(X_n)), \hat{+}\right)$ is a semigroup.

Proof We show that the binary operation $\hat{+}$ is associative. For this, let A, B, C be arbitrary elements in $P(W_{\tau_n}^{S(\overline{n},Y)}(X_n))$. By the superassociativity of the operation $\hat{S}^n$, we have $(A\hat{+}B)\hat{+}C = \hat{S}^n(\hat{S}^n(A, B, \dots, B), C, \dots, C)$ and

$$A\hat{+}(B\hat{+}C) = A\hat{+}\hat{S}^n(B, C, \dots, C)$$
$$= \hat{S}^n(A, \hat{S}^n(B, C, \dots, C), \dots, \hat{S}^n(B, C, \dots, C)).$$

As a result, $(A\hat{+}B)\hat{+}C = A\hat{+}(B\hat{+}C)$. Thus, $\left(P(W_{\tau_n}^{S(\overline{n},Y)}(X_n)), \hat{+}\right)$ is a semigroup.

The obtained semigroup in Theorem 5 is called the *diagonal power semigroup* derived from the power Menger algebra $\left(P(W_{\tau_n}^{S(\overline{n},Y)}(X_n)), \hat{S}^n\right)$.

Let A and B be subsets of full terms with an invariant set. For each $i = 1, \dots, n$, the binary operation $\cdot_{\{x_i\}} : P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right) \times P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right) \to P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)$ can be defined by $A \cdot_{\{x_i\}} B = \hat{S}^n(A, \{x_1\}, \dots, \{x_{i-1}\}, B, \{x_{i+1}\}, \dots, \{x_n\})$.

The fact that the binary operation $\cdot_{\{x_i\}}$ satisfies the associative law is proved in the following theorem.

Theorem 6 $(P(W_{\tau_n}^{S(\overline{n},Y)}(X_n)), \cdot_{\{x_i\}})$ is a semigroup.

Proof To show that the binary operation $\cdot_{\{x_i\}}$ is associative, let A, B, C be sets of $S(\overline{n}, Y)$-full terms. Due to the satisfaction of the operation $\hat{S}^n$ of a superassociative law over the set $P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)$, we obtain $\left(A \cdot_{\{x_i\}} B\right) \cdot_{\{x_i\}} C = \hat{S}^n(A, \{x_1\}, \dots, \{x_{i-1}\}, B, \{x_{i+1}\}, \dots, \{x_n\}) \cdot_{\{x_i\}} C = A \cdot_{\{x_i\}} \hat{S}^n(B, \{x_1\}, \dots, \{x_{i-1}\}, C, \{x_{i+1}\}, \dots, \{x_n\})$, which equals $A \cdot_{\{x_i\}} (B \cdot_{\{x_i\}} C)$. Therefore, we conclude that $(P(W_{\tau_n}^{S(\overline{n},Y)}(X_n)), \cdot_{\{x_i\}})$ forms a semigroup.

On the Cartesian product $P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n$ of tree languages full terms of type τ_n, the binary operation, denoted by $\hat{*}$, is defined by

$$(A_1, \dots, A_n) \hat{*} (B_1, \dots, B_n)$$
$$= (\hat{S}^n(A_1, B_1, \dots, B_n), \dots, \hat{S}^n(A_n, B_1, \dots, B_n))$$

for all $(A_1, \dots, A_n), (B_1, \dots, B_n) \in P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n$. As a result, we prove:

Theorem 7 $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n, \hat{*}\right)$ is a semigroup.

Proof First, we let $(A_1, \dots, A_n), (B_1, \dots, B_n), (C_1, \dots, C_n) \in P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n$. Because a non-deterministic superposition operation satisfies the superassociative law, the equation $((A_1, \dots, A_n) \hat{*} (B_1, \dots, B_n)) \hat{*} (C_1, \dots, C_n) = (A_1, \dots, A_n) \hat{*} ((B_1, \dots, B_n) \hat{*}

$(C_1, \ldots, C_n))$ is directly obtained. Hence, $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n, \hat{*}\right)$ is a semigroup.

Results and discussion

This section is contributed to several embedding theorems of semigroups derived from the algebra of $S(\overline{n}, Y)$-full terms into semigroups of tree languages of $S(\overline{n}, Y)$-full terms.

Theorem 8 The following statements hold:

(1) The diagonal semigroup $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), +\right)$ can be embedded into the binary comitant $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n, *\right)$.

(2) The power diagonal semigroup $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right), \hat{+}\right)$ can be embedded into $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n, \hat{*}\right)$.

Proof We first prove the statement (1). For any $S(\overline{n}, Y)$-full term t of type τ_n, the mapping $\sigma: W_{\tau_n}^{S(\overline{n},Y)}(X_n) \rightarrow W_{\tau_n}^{S(\overline{n},Y)}(X_n)$ can be defined by $\sigma(t) = (\underbrace{t, \ldots, t}_{n \text{ times}})$. It is obvious that σ is an injection. To show that σ is a homomorphism, we let $s, t \in W_{\tau_n}^{S(\overline{n},Y)}(X_n)$. By this definition, we have that $\rho(s + t) = \rho(S^n(s, t, \ldots, t)) = (\underbrace{S^n(s, t, \ldots, t), \ldots, S^n(s, t, \ldots, t)}_{n})$ $= (s, \ldots, s) + (t, \ldots, t) = \rho(s) + \rho(t)$. As a consequence, we say that $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), +\right)$ can be embedded into $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n, *\right)$. To show (2), we define the mapping $\vartheta: P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right) \rightarrow P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n$ by $\vartheta(D) = (\underbrace{D, \ldots, D}_{n \text{ times}})$ for all $D \in P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)$. Clearly, ϑ is injective. Furthermore, we have $\vartheta(A \hat{+} B) = \vartheta(A) \hat{+} \vartheta(B)$ because $\vartheta(A \hat{+} B) = \vartheta\left(\hat{S}^n(A, B_1, \ldots, B_n)\right)$ and that every position of the tuple is equal to $\hat{S}^n(A, B_1, \ldots, B_n)$. On the other hand, we have $(A, \ldots, A) \hat{+} (B, \ldots, B)$, which is equal to $\left(\hat{S}^n(A, B_1, \ldots, B_n), \ldots, \hat{S}^n(A, B_1, \ldots, B_n)\right)$. We conclude that the mapping ϑ is a monomorphism from $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right), \hat{+}\right)$ to the semigroup $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n, \hat{*}\right)$.

Finally, the facts that the semigroups of $S(\overline{n}, Y)$-full terms can be embedded into the semigroups of tree languages of $S(\overline{n}, Y)$-full terms will be proved.

Theorem 9 The following statements are obtained:

(1) The Menger algebra $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), S^n\right)$ is embeddable into the power Menger algebra $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right), \hat{S}^n\right)$,

(2) the semigroup $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), +\right)$ is embeddable into the power semigroup $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right), \hat{+}\right)$,

(3) the semigroup $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), \cdot_{x_i}\right)$ is embeddable into $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right), \cdot_{\{x_i\}}\right)$,

(4) the semigroup $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n, *\right)$ is embeddable into $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n, \hat{*}\right)$.

Proof We first show that the statement (1) holds. To do this, we define the mapping $\eta : \left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), S^n\right) \to \left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right), \hat{S}^{\boldsymbol{n}}\right)$ by $\eta(t) = \{t\}$ for all $S(\overline{n},Y)$-full term t of type τ_n. Obviously, η is an injective mapping. Moreover, it has a homomorphism property, i.e., $\eta(S^n(t, s_1, \dots, s_n)) = \hat{S}^n(\eta(t), \eta(s_1), \dots, \eta(s_n))$. In fact, we give a proof by induction on the complexity of an $S(\overline{n},Y)$-full term t. If $t = f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ where $\alpha \in S(\overline{n},Y)$, we have

$$\eta\left(S^n\left(f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right), s_1, \dots, s_n\right)\right) = \eta\left(f_i\left(s_{\alpha(1)}, \dots, s_{\alpha(n)}\right)\right) = \left\{f_i\left(s_{\alpha(1)}, \dots, s_{\alpha(n)}\right)\right\}$$

and

$$\hat{S}^n\left(\eta\left(f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right)\right), \eta(s_1), \dots, \eta(s_n)\right) = \hat{S}^n\left(\left\{f_i\left(x_{\alpha(1)}, \dots, x_{\alpha(n)}\right)\right\}, \{s_1\}, \dots, \{s_n\}\right) = \left\{f_i\left(s_{\alpha(1)}, \dots, s_{\alpha(n)}\right)\right\}.$$

If $t = f_i(t_1, \dots, t_n)$ and assume now that $\eta\left(S^n(t_j, s_1, \dots, s_n)\right) = \hat{S}^n\left(\eta(t_j), \eta(s_1), \dots, \eta(s_n)\right)$ for every $1 \leq j \leq n$, consequently, we have

$$\eta\left(S^n\left(f_i(t_1, \dots, t_n), s_1, \dots, s_n\right)\right) = \{S^n(f_i(t_1, \dots, t_n), s_1, \dots, s_n)\} = \{f_i(S^n(t_1, s_1, \dots, s_n), \dots, S^n(t_n, s_1, \dots, s_n))\}$$

and

$$\hat{S}^n\left(\eta(f_i(t_1, \dots, t_n)), \eta(s_1), \dots, \eta(s_n)\right) = \hat{S}^n(\{f_i(t_1, \dots, t_n)\}, \{s_1\}, \dots, \{s_n\}) = \left\{f_i(r_1, \dots, r_n) \mid r_j \in \hat{S}^n(\{t_j\}, \{s_1\}, \dots, \{s_n\})\right\}.$$

Thus, $\eta(S^n(f_i(t_1, \dots, t_n), s_1, \dots, s_n))$ and the set $\hat{S}^n\left(\eta(f_i(t_1, \dots, t_n)), \eta(s_1), \dots, \eta(s_n)\right)$ are the same thing. Therefore, η is a monomorphism from the algebra $\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n), S^n\right)$ to the power Menger algebra $\left(P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right), \hat{S}^{\boldsymbol{n}}\right)$. To prove the statements (2) and (3), we define the mapping $\beta : W_{\tau_n}^{S(\overline{n},Y)}(X_n) \to P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)$ by $\beta(t) = \{t\}$ for all $t \in W_{\tau_n}^{S(\overline{n},Y)}(X_n)$. Clearly, β is injective. It is not difficult to verify that the following equations hold: $\beta(s + t) = \beta(s) \hat{+} \beta(t)$ and $\beta(s \cdot_{x_i} t) = \beta(s) \cdot_{\{x_i\}} \beta(t)$. Finally, we prove that (4) holds. Let $(t_1, \dots, t_n)$ be an n-tuple of the Cartesian set $W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n$. The mapping $\varphi : W_{\tau_n}^{S(\overline{n},Y)}(X_n)^n \to P\left(W_{\tau_n}^{S(\overline{n},Y)}(X_n)\right)^n$ can be naturally defined by $\varphi((t_1, \dots, t_n)) = (\{t_1\}, \dots, \{t_n\})$. It is clear that φ is injective. The proof of a homomorphism property is omitted.

We give a final remark that this paper deals with the study of semigroups of full terms obtained from full transformations with an invariant set. The results given in Theorems 1, 2 and 3 show that the set of full terms with an invariant set is closed under application of the superposition S^n and each binary operation defined in a different approach is associative. Besides, in Theorems 4, 5 and 6, sets of these full terms are considered in a canonical way. From these, a close connection among these semigroups is mentioned in Theorems 8 and 9. The importance of such theorems is now summarized. If some properties of the semigroups given in Theorems 1, 2 and 3 are complicated and a process to achieve them is difficult, we can transfer a study to another structure which is connected by the concept of homomorphisms and return to answer the original problem in the first one.

In terms of development of research in this direction, our results can be viewed as a continuation of the work of (Wattanatripop, & Changphas, 2021b, Phuapong, & Pookpienlert, 2022) and other related topics in this direction in terms of a construction of algebraic structures which have many properties. This means that although the concept of full terms with an invariant set was defined, the operation considered on such full terms is only a superposition S^n of type $n+1$ for some positive integer n allowing a construction of the algebra satisfying the axiom of superassociativity. To fulfill this gap, this paper determines many binary operations defined for such terms and presents their algebraic properties in more complicated structures.

Conclusion

It can be noticed here that all semigroups defined on the set of all full terms with an invariant set can be represented by the semigroups of their corresponding languages by considering an image of a singleton set of full terms with an invariant set in a natural way. We conclude this paper with a few open problems and possible directions for further research. Characterizations for any element in some semigroups mentioned in Theorems 1, 2 and 3 to be an idempotent element, and a regular element remain open. Moreover, an extension of semigroups of full terms with an invariant set to ternary semigroups of full terms with an invariant set which is more complicated than semigroups is also interesting. Hopefully, these questions will be answered in the near future.

Acknowledgement

This work was supported by Rajamangala University of Technology Rattanakosin, Thailand.

References

Chansuriya, N. (2021). All maximal idempotent submonoids of generalized cohypersubstitutions of type $\tau = (2)$. *Discussiones Mathematicae - General Algebra and Applications*, *41*(1), 45-54.

Chinram, R., & Baupradist, S. (2019). Magnifying elements of semigroups of transformations with invariant set. *Asian-European Journal of Mathematics*, *12*(4), 1950056.

Denecke, K. (2016). The partial clone of linear terms. *Siberian Mathematical Journal*, *57*(4), 589-598.

Denecke, K. (2022). Partial Menger algebras and their weakly isomorphic representation. *Mathematics Open*, *1*, 2250003.

Denecke, K., & Hounnon, H. (2021). Partial Menger algebras of terms. *Asian-European Journal of Mathematics*, *14*(6), 2150092.

Denecke, K., & Jampachon, P. (2003). Clones of full terms. *Algebra and Discrete Mathematics*, *4*, 1-11.

Dudek, W. A., & Trokhimenko, V. S. (2021). Menger algebras of k-commutative n-place functions. *Georgian Mathematical Journal*, *28*(3), 355-361.

Honyam, P., & Sanwong, J. (2011). Semigroups of transformations with invariant set. *Journal of the Korean Mathematical Society*, *48*(2), 289-300.

Joomwong, J., & Phusanga, D. (2021). On Green's relations which are related to an algebraic system of Type (m,n). *Southeast Asian Bulletin of Mathematics*, *45*(6), 897-904.

Kumduang, T., & Sriwongsa, S. (2022). A note on superassociative algebra of terms determined by singular mappings on a finite set. *International Journal of Mathematics and Computer Science*, *14*(4), 1541-1546.

Lekkoksung, N., & Lekkoksung, S. (2021). On partial clones of k-terms. *Discussiones Mathematicae - General Algebra and Applications*, *41*(2), 361-379.

Phuapong, S., & Kumduang, T. (2021). Menger algebras of terms induced by transformations with restricted range. *Quasigroups and Related Systems*, *29*, 255-268.

Phuapong, S., & Pookpienlert, C. (2022). $S(n_i, Y_i)$-terms and their algebraic properties. *Thai Journal of Mathematics*, *20*(1), 337-346.

Salehia, S., & Steinby, M. (2007). Tree algebras and varieties of tree languages. *Theoretical Computer Science*, *377*, 1-24.

Sarkar, M., & Singh, S. N. (2022). On certain semigroups of transformations with an invariant set. *Asian-European Journal of Mathematics*, *15*(11), 2250198.

Wattanatripop, K., & Changphas, T. (2019). The clone of $K^*(n,r)$-full terms. *Discussiones Mathematicae - General Algebra and Applications, 39*(2), 277-288.

Wattanatripop, K., & Changphas, T. (2021a). On $nd-K^*(n,r)$-Full hypersubstitutions. *Discussiones Mathematicae - General Algebra and Applications*, *41*(2), 213-227.

Wattanatripop, K., & Changphas, T. (2021b). The Menger algebra of terms induced by order-decreasing Transformations. *Communications in Algebra*, *49*(7), 3114-3123.