The solution of Diophantine equation $48^x + 84^y = z^2$

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Abstract

In this paper, the researchers studied the solutions of Diophantine equation $48^x + 84^y = Z^2$ where x, y and z were non-negative integers. The researchers showed that the Diophantine equation $48^x + 84^y = Z^2$ had a unique non-negative integers solution $(x, y, z) = (1, 0, 7)$.

Keywords: exponential Diophantine equation, Catalan's conjecture, quadratic congruence

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Introduction

The Diophantine equations of type $a^x + b^y = c^z$ introduced by Cao (1987). He proved that this equation had at the most one solution under certain conditions. In 2005, Acu (2005) studied Diophantine equations of type $a^x + b^y = c^z$ for primes a and b. He studied the solutions of three kinds of Diophantine equations: (i) the equation of type $p^x + p^y = p^z$, where p was the prime number, (ii) the equation of type $p^x + p^y = (2p)^z$, where p was the prime number and (iii) the equation of type $p^x + q^y = (pq)^z$, where p and q were given the prime numbers. In 2007, Acu (2007) studied Diophantine equations $2^x + 5^y = z^2$. He found that the previous Diophantine equation had only two solutions in non-negative integers $(x, y, z) \in \{(3,0,3),(2,1,3)\}$. In 2008, Pumnea and Nicoar (2008) studied Diophantine equations of the form $a^x + b^y = z^2$, for example: $2^x + 7^y = z^2, 2^x + 11^y = z^2$ and $2^x + 13^y = z^2$. In 2011, Suvarnamani (2011) studied Diophantine equation $2^x + p^y = z^2$ where p was prime number which was more than 2, he found that (i) for each prime number p, this equation had a solution $(x, y, z) = (3,0,3)$, (ii) for $p = 3$, the Diophantine equation this equation had a solution $(x, y, z) = (4,2,5)$ and (iii) for $p = 1 + 2^{k+1}$ where k was non-negative integer, this equation had a solution $(x, y, z) = (2k,1,1+2k)$. In 2011, Suvarnamani *et al*. (2011) studied two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. They found that both of them had no solution in non-negative integers. In 2012-2013, Sroysang (2012a, 2012b, 2012c, 2013, 2014) published series of papers in relation to the Diophantine equation $a^x + b^y = c^z$. In 2013, Rabago (2013a, 2013b, 2013c, 2013d, 2013e) gave all solutions to several Diophantine equations of type $p^x + q^y = z^2$ for examples: $5^x + 31^y = z^2$, $7^x + 29^y = z^2$, $13^x + 23^y = z^2$, $47^x + 97^y = z^2$, $61^x + 83^y = z^2$, $3^x + 19^y = z^2$, $3^x + 91^y = z^2$, $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$. He found that: (1) $(x, y, z) = (1,1,6)$ was non-negative integer solution of the Diophantine equations $5^x + 31^y = z^2$, $5^x + 31^y = z^2$ and $13^x + 23^y = z^2$, (2) $(x, y, z) = (1,1,12)$ was non-negative integer solution of the Diophantine equations $47^x + 97^y = z^2$ and $61^x + 83^y = z^2$, (3) the Diophantine equation $3^x + 19^y = z^2$ had exactly two solutions $(x, y, z) \in \{(1,0,2),(4,1,10)\}$ in non-negative integers, (4) the Diophantine equation $3^x + 91^y = z^2$ had exactly two solutions $(x, y, z) \in \{(1,0,2),(2,1,10)\}$ in non-negative integers and (5) the Diophantine equations $17^x + 19^y = z^2$ had a unique solution $(x, y, z) = (1,1,6)$ and $71^x + 73^y = z^2$ had a unique solution $(x, y, z) = (1,1,12)$ in non-negative integers, respectively. In 2014, Sroysang (2014) studied the Diophantine equation $46^x + 64^y = z^2$. He found that the Diophantine equation $46^x + 64^y = z^2$ had no non-negative integer solutions. Furthermore, he posed some open problem about the Diophantine equation in the form $m^x + n^y = z^2$ when $m = 10a + b,\ n = 10b + a$ and $a, b \in \{0,1,2,\ldots,9\}$.

In this paper, the researchers studied Diophantine equation $48^x + 84^y = z^2$ where x, y and z were non-negative integers.

Preliminaries

Let p was an odd prime and a was a positive integer where $\gcd(a, p) = 1$. If the quadratic congruence $x^2 \equiv a \pmod{p}$ had a solution, then a was said to be a quadratic residue of p. Otherwise, a was called a quadratic non-residue of p. Introduced the Legendre symbol $\left(\frac{a}{p}\right)$ which was defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 \ ; & \text{If } a \text{ is a quadratic residue of } p, \\ -1 \ ; & \text{If } a \text{ is a quadratic non-residue of } p \end{cases}$$

In this paper, Catalan's conjecture (Mihailescu, 2004), which stated that the only solution in integers $a > 1$, $b > 1$, $x > 1$ and $y > 1$ of the equation $a^x - b^y = 1$ is $(a, b, x, y) = (3, 2, 2, 3)$, was used. Moreover, the researchers needed the following well-known facts about the Legendre's symbol.

Theorem 2.1 (David, 2007) Let p was an odd prime and a and b be integers which were relatively prime to p. Then the Legendre symbol had the following properties:

(1) If $a \equiv b \pmod{p}$, then $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$

(2) $\left(\frac{a^2}{p}\right) = 1$

(3) $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$

(4) $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$

(5) $\left(\frac{1}{p}\right) = 1$ and $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$.

Proof. See (David, 2007).

Corollary 2.2 (David, 2007) If p was an odd prime, then

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 \ ; & \text{if } p \equiv 1 \pmod{4} \\ -1 \ ; & \text{if } p \equiv 3 \pmod{4} \end{cases}.$$

Proof. See (David, 2007).

Results

Lemma 3.1 The Diophantine equation $48^x + 1 = z^2$ had a unique non-negative integers solution $(x, z) = (1, 7)$.

Proof. Let x and z were non-negative integers. Clearly, the lemma was true when $x = 0$. If $z = 0$, then the Diophantine equation $48^x + 1 = z^2$ becomes: $48^x + 1 = 0$. There were no non-negative integers x satisfy the previous equation. Then we would consider in case $x, z > 0$ by divided into two cases:

Case 1 If $x = 1$, then Diophantine equation $48^x + 1 = z^2$ becomes $z^2 = 49$. Since z was positive integer, then $z = 7$, then we got that $(x, z) = (1, 7)$ was a non-negative integers solution of Diophantine equation $48^x + 1 = z^2$.

Case 2 If $x > 1$, we saw that $48^x = z^2 - 1 = (z-1)(z+1)$. Since $48^x = 1(48^x)$ $= (2^x)(24^x) = (3^x)(16^x) = (4^x)(12^x) = (6^x)(8^x)$. We would consider following twelve sub-cases:

Sub-case 2.1 If $z - 1 = 1$ and $z + 1 = 48^x$, then $48^x - 1 = 2$. So $48^x = 3$. It was impossible.

Sub-case 2.2 If $z - 1 = 48^x$ and $z + 1 = 1$, then $1 - 48^x = 2$. So $48^x = -1$. It was impossible because LHS was negative integer but RHS was positive integer.

Sub-case 2.3 If $z - 1 = 48^u$ and $z + 1 = 48^{x-u}$ where $u \in \mathbb{N} \cup \{0\}$ and $x > 2u$, then $48^{x-u} - 48^u = 2$. The researchers got that $48^u\left(48^{x-2u} - 1\right) = 2$. We would consider following two sub-cases:

Sub-case 2.3.1 If $48^u = 1$ and $48^{x-2u} - 1 = 2$, then $u = 0$ and $48^x - 1 = 2$. We got that $48^x = 3$. It was impossible because that there were no positive integer x such that $48^x = 3$.

Sub-case 2.3.2 If $48^u = 2$ and $48^{x-2u} - 1 = 1$, then $48^{x-2u} = 2$. It was impossible because that there were no positive integer $x - 2u$ such that $48^{x-2u} = 2$.

Sub-case 2.4 If $z - 1 = 48^u$ and $z + 1 = 48^{x-u}$ where $u \in \mathbb{N} \cup \{0\}$ and $x \le 2u$, then $48^{x-u} - 48^u = 2$. We got that $48^u\left(48^{x-2u} - 1\right) = 2$. Then LHS was zero or negative integer but RHS was positive integer.

Sub-case 2.5 If $z - 1 = 2^x$ and $z + 1 = 24^x$, then $24^x - 2^x = 2$. So $24^x = 2(1 + 2^{x-1})$.

We got $2^{x-1}12^x = 1+2^{x-1}$. It was impossible because LHS was even but RHS was odd.

Sub-case 2.6 If $z-1=24^x$ and $z+1=2^x$, then $2^x - 24^x = 2$. It was impossible because LHS was negative integer but RHS was positive integer.

Sub-case 2.7 If $z-1=3^x$ and $z+1=16^x$, then $16^x - 3^x = 2$. So $16^x = 2+3^x$. It was impossible because LHS was even but RHS was odd.

Sub-case 2.8 If $z-1=16^x$ and $z+1=3^x$, then $3^x - 16^x = 2$. It was impossible because LHS was negative integer but RHS was positive integer.

Sub-case 2.9 If $z-1=4^x$ and $z+1=12^x$, then $12^x - 4^x = 2$. So $12^x = 2+4^x$. We got $2^{x-1}6^x = 2^{2x-1}+1$. It was impossible because LHS was even but RHS was odd.

Sub-case 2.10 If $z-1=12^x$ and $z+1=4^x$, then $4^x - 12^x = 2$. It was impossible because LHS was negative integer but RHS was positive integer.

Sub-case 2.11 If $z-1=6^x$ and $z+1=8^x$, then $8^x - 6^x = 2$. Then $x=1$. So $z=7$. Therefore $(x,z)=(1,7)$ was a solution of Diophantine equation $48^x+1=z^2$.

Sub-case 2.12 If $z-1=8^x$ and $z+1=6^x$, then $6^x - 8^x = 2$. It was impossible because LHS was negative integer but RHS was positive integer.

From case 1 and case 2 , the Diophantine equation $48^x+1=z^2$ had a unique non-negative integers solution $(x,z)=(1,7)$.

Lemma 3.2 The Diophantine equation $84^x+1=z^2$ had no non-negative integers solution.

Proof. Let x and z be non-negative integers. Clearly, the lemma was true when $x=0$.
If $z=0$, then the Diophantine equation $84^x+1=z^2$ becomes: $84^x+1=0$. There were no non-negative integers x satisfy this equation. Then we would consider in case $x,z>0$ by divided into two cases:

Case 1 If $x=1$, then Diophantine equation $84^x+1=z^2$ becomes: $z^2=85$. There were no positive integers z satisfied this equation.

Case 2 If $x > 1$, then $84^x = z^2 - 1 = (z-1)(z+1)$. Since $84^x = 1(84^x) = (2^x)(41^x)$. We would consider in six subcases:

Sub-case 2.1 If $z-1=1$ and $z+1=84^x$, then $84^x - 1 = 2$. So $84^x = 3$. It was impossible.

Sub-case 2.2 If $z-1=84^x$ and $z+1=1$, then $1-84^x = 2$. So $84^x = -1$. It was impossible because LHS was negative integer but RHS was positive integer.

Sub-case 2.3 If $z-1=84^u$ and $z+1=84^{x-u}$ where $u \in \mathbb{N} \cup \{0\}$ and $x > 2u$. Then $84^{x-u} - 84^u = 2$. We saw that $84^u\left(82^{x-2u}-1\right)=2$. We would consider two sub-cases:

Sub-case 2.3.1 If $84^u = 1$ and $84^{x-2u} - 1 = 2$, then $u = 0$ and $84^x - 1 = 2$. We had that $84^x = 3$. It was impossible.

Sub-case 2.3.2 If $84^u = 2$ and $84^{x-2u} - 1 = 1$, then $84^{x-2u} = 2$. It was impossible.

Sub-case 2.4 If $z-1=84^u$ and $z+1=84^{x-u}$ where $u \in \mathbb{N} \cup \{0\}$ and $x \le 2u$, then $84^{x-u} - 84^u = 2$. We got that $84^u\left(82^{x-2u}-1\right)=2$. Since $x \le 2u$, then $82^{x-2u} - 1$ was zero or negative integer. So $84^u\left(82^{x-2u}-1\right)$ was also zero or negative integer contradiction to 2 was positive integer.

Sub-case 2.5 If $z-1=2^x$ and $z+1=41^x$, then $41^x - 2^x = 2$. So $41^x = 2 + 2^x$. It was impossible because LHS was an odd but RHS was an even.

Sub-case 2.6 If $z-1=24^x$ and $z+1=2^x$, then $2^x - 41^x = 2$. It was impossible because LHS was negative integer but RHS was positive integer.

From case 1 and case 2, the Diophantine equation $84^x + 1 = z^2$ had no non-negative integers solution.

Theorem 3.3 The Diophantine equation $48^x + 84^y = z^2$ had a unique non-negative integers solution $(x, y, z) = (1, 0, 7)$.

Proof. Let x, y and z were non-negative integers. If $z = 0$, then the Diophantine equation $48^x + 84^y = z^2$ became: $48^x + 84^y = 0$. There were no non-negative integers x, y satisfy this equation. Now, let $z > 0$ and we would consider in three cases:

Case 1 If $x = 0$, then the Diophantine equation $48^x + 84^y = z^2$ could be written as $84^y + 1 = z^2$. By Lemma 3.2, this equation had no solution in non-negative integers.

Case 2 If $y=0$, then the Diophantine equation $48^x+84^y=z^2$ could be written as $48^x+1=z^2$. By Lemma 3.1, this equation had only one solution $(x,z)=(1,7)$ in non-negative integers. Then $(x,y,z)=(1,0,7)$ was a non-negative integers solution of Diophantine equation $48^x+84^y=z^2$.

Case 3 If $xy>0$, then we considered in two sub-cases:

Sub-case 3.1 If x was even, then $x=2k$ for some positive integer k. Then the Diophantine equation $48^x+84^y=z^2$ could be written as $48^{2k}+84^y=z^2$ or $84^y=(z-48^k)(z+48^k)$. If $z-48^k=84^u$ and $z+48^k=84^{y-u}$ where u was non-negative integer and $y>2u$, then $84^{y-u}-84^u=(z+48^k)-(z-48^k)=2(48^k)$. So $84^u(84^{y-2u}-1)=2(48^k)$. Since $48^k=(2^4\cdot 3)^k=2^{4k}3^k$, then $2(48^k)=2^{4k+1}3^k$. So $84^u(84^{y-2u}-1)=2^{4k+1}3^k$. We would consider in following sub-cases:

Sub-case 3.1.1 If $84^u=1$ and $84^{y-2u}-1=2^{4k+1}3^k$, we got that $u=0$ and $84^y-1=2^{4k+1}3^k$. This was impossible because 84^y-1 was an odd but $2^{4k+1}3^k$ was an even.

Sub-case 3.1.2 If $84^u=2^s$ where $1\le s\le 4k+1$ and $84^{y-2u}-1=3^k$. Then the equation $84^{y-2u}-1=3^k$ became: $84^{y-2u}-3^k=1$. This equation was a Diophantine equation by Catalan's type $a^x-b^y=1$. It had no integers solution when $\min\{84,3,y-2u,k\}>1$. If $k=1$, then the equation $84^{y-2u}-3^k=1$ became: $84^{y-2u}=4$. It was impossible. If $y-2u=1$, then $84-3^k=1$. So $3^k=83$. It was impossible.

Sub-case 3.1.3 If $84^u=2^{4k+1}(3^s)$ where $1\le s<k$ and $84^{y-2u}-1=3^{k-s}$. We got similar result to sub-case 3.1.2.

Sub-case 3.1.4 If $84^u=3^{k-s}$ and $84^{y-2u}-1=2^{4k+1}(3^s)$ where $1\le s<k$. This was impossible because $84^{y-2u}-1$ was an odd but $2^{4k+1}(3^s)$ was an even.

Sub-case 3.1.5 If $84^u=3^k$ and $84^{y-2u}-1=2^s$ where $1\le s\le 4k+1$. Then the equation $84^{y-2u}-1=2^s$ became $84^{y-2u}-2^s=1$. This equation was a Diophantine equation by Catalan's type $a^x-b^y=1$. It had no solution when $\min\{84,2,y-2u,s\}>1$. If $s=1$, then $84^{y-2u}=3$. It was impossible. If $y-2u=1$, then $2^s=83$. It was impossible.

Sub-case 3.1.6 If $84^u = 2^{4k+1}3^k$ and $84^{y-2u} - 1 = 1$, we get that $84^{y-2u} = 2$. This is impossible.

If $z - 48^k = 84^u$ and $z + 48^k = 84^{y-u}$ where u was non-negative integer and $y \le 2u$, then $84^{y-u} - 84^u = (z+48^k) - (z-48^k) = 2(48^k)$. So $84^u(84^{y-2u} - 1) = 2(48^k)$. Since $y \le 2u$, then $84^u(84^{y-2u} - 1) \le 0$. This was impossible because LHS was zero or negative integer but RHS was positive integer.

Sub-case 3.2 If x is odd, then $x=2k+1$ for some non-negative integer k. Then $48^x + 84^y = z^2$ became: $48^{2k+1} + 84^y = z^2$. Since $48^{2k+1} \equiv -1 \pmod 7$ and $84^y \equiv 0 \pmod 7$, then $48^{2k+1} + 84^y \equiv -1 \pmod 7$. So $z^2 \equiv -1 \pmod 7$. Since 7 was prime and $7 \equiv 3 \pmod 4$, by Theorem 2.1 and Lemma 2.2, we had $\left(\frac{-1}{7}\right) = (-1)^{\frac{7-1}{2}} = -1$, then $z^2 \equiv -1 \pmod 7$ had no integer solutions. Therefore $48^x + 84^y = z^2$ had no integers solution.

From case 1, case 2 and case 3, we concluded that the Diophantine equation $48^x + 84^y = z^2$ had a unique non-negative integers solution $(x, y, z) = (1, 0, 7)$.

Conclusion

In this paper, we showed that the Diophantine equation $48^x + 84^y = z^2$ had a unique non-negative integers solution $(x, y, z) = (1, 0, 7)$.

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References

Acu, D. 2005. On a Diophantine equations of type $a^x + b^y = c^z$. *Gen. Math. 13*: 67-72.

Acu, D. 2007. On a Diophantine equation $2^x + 5^y = z^2$. *Gen. Math. 15*: 145-148.

Cao, Z. 1987. On the Diophantine equation $a^x + b^y = c^z$. *Chinesse Sc. Bull. 32*: 1519-1521 (in Chinese).

David, M. B. 2007. Elementary number theory. 6th ed., McGraw-Hill, Singapore.

Mihailescu, P. 2004. Primary cycolotomic units and a proof of Catalan's conjecture. *J. Reine Angew. Math. 27*: 167-95.

Pumnea, C.E. and A.D. Nicoar. 2008. On a Diophantine equation of $a^x + b^y = z^2$. *Educatia Matematica. 4*: 65-75.

Rabago, J.F.T. 2013a. On an open problem by B. Sroysang. *Konuralp Journal of Mathematics.1*: 30-32.

Rabago, J.F.T. 2013b. A note on an open problem by B. Sroysang. *Sci. Technol. RMUTT J. 3*: 41-43.

Rabago, J.F.T. 2013c. More on Diophantine equations of type $p^x + q^y = z^2$. *International Journal of Mathematics and Scientific Computing 3*: 15-16.

Rabago, J.F.T. 2013d. On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$.*International Journal of Mathematics and Scientific Computing 3*: 28-29.

Rabago, J.F.T. 2013e. On two Diophantine equations $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$. *Mathematical Journal of Interdisciplinary Sciences 2*: 19-24.

Suvarnamani, A. 2011. Solutions of the Diophantine equation $2^x + p^y = z^2$. *International Journal of Mathematical Sciences and Applications 1*: 1415-1419.

Suvarnamani, A., A. Singta and S. Chotchaisthit. 2011. On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$. *Sci. Technol. RMUTT J. 1*: 25-28.

Sroysang, B. 2012a. More on the Diophantine equation $8^x + 19^y = z^2$. *Int. J. Pure Appl. Math. 81*: 601-604.

Sroysang, B. 2012b. On the Diophantine equation $31^x + 32^y = z^2$. *Int. J. Pure Appl. Math. 81*: 609-612.

Sroysang, B. 2012c. On the Diophantine equation $3^x + 5^y = z^2$. *Int. J. Pure Appl. Math. 81*: 605-608.

Sroysang, B. 2013. On the Diophantine equation $3^x + 17^y = z^2$. *Int. J. Pure Appl. Math. 89*: 111-114.

Sroysang, B. 2014. On the Diophantine equation $46^x + 64^y = z^2$. *Int. J. Pure Appl. Math. 91*: 399-402.