Distributional solutions of *n*th-order differential equations of the Bessel equation

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Abstract

In this paper, we study the distributional solutions of *n*th-order differential equation of the form $x^2 y^{(n)}(x) + xy^{(n-1)}(x) + (b^2x^2 - p^2)y^{(n-2)}(x) = 0$, where $p \geq 0, n \geq 2, b > 0$ and x is a real variable. These solutions are obtained in the form of infinite series of the Dirac delta functions and its derivatives. We employ these solutions to observe their interesting features.

Keywords: Dirac delta function, distributional solution, Bessel equation

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Introduction

This paper, we study distributional solution of the ordinary differential equations and the functional differential equations in various spaces of generalized functions. Methods of the distribution theory have been used in the several important areas, such as theoretical and mathematical physics, theory of the partial differential equations, the functional analysis and etc. It is well known that the normal linear homogeneous system of the ordinary differential equation with infinitely smooth coefficients has been in the classical only.

In 1982, Wiener (1982) had studied the various differential equations with singular coefficients, and obtained their distributional solutions. After that in 1983, Shah (1983) surveyed the work in this field and had exhibited a unified approach in the study of both distributional and entire solutions to some classes of the linear ordinary differential equations.

In 1987, Littlejohn and Kanwal (1987) studied the distributional solutions to the hyper geometric differential equation. These solutions were obtained in the form of infinite series of the Dirac delta functions and its derivatives. Another motivation for studying solutions of the form of infinite series of the Dirac delta functions and its derivatives to ordinary differential equations comes from the works of (Morton and krall, 1978; Krall, 1981; Cooke and Wiener, 1984; Littlejohn, 1984; Wiener and Cooke, 1990; Wiener *et al*., 1991; Hernandez-Urena and Estrada, 1995). These researchers had collectively shown that weight distributions for a certain class of orthogonal polynomials have the form of infinite series of the Dirac delta functions and its derivatives, and simultaneously satisfy a system of ordinary differential equations.

In 1998, Kanwal (1983) studied the solutions of the Bessel equation

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2) y(x) = 0. \qquad (1)$$

He found that such equation has a distributional solution.

In this paper, our aim is to present the solutions of the form

$$\sum_{k=0}^{\infty} a_k \delta^{(k)}(x) \qquad (2)$$

for *n*th-order differential equation of the form

$$x^2 y^{(n)}(x) + xy^{(n-1)}(x) + (b^2 x^2 - p^2) y^{(n-2)}(x) = 0, \qquad (3)$$

where $p \geq 0, n \geq 2, b > 0$ and x is a real variable. In addition to deriving these solutions for their intrinsic value, we want to display their uses and exhibit their interplay with related results in the theory of ordinary differential equations. For instance, if we let $n = 2$, and $b^2 = 1$ then (3) is reduced to (1).

Methodology

In this section, we present the basic operations that can be applied to a series of Dirac delta functions. We follow the standard notations concerning the usual spaces of generalized functions D', E' and S' (Schwartz, 1957; Schwartz,1959; Kanwal, 1983).

Let us first recall that the delta functions $\delta^{(k)}(x)$ are distributions of the standard space $D'(\mathbb{R})$, defined by

$$\left\langle \delta^{(k)}(x), \phi(x) \right\rangle = (-1)^k \phi^{(k)}(0), \tag{4}$$

provided that ϕ is smooth. The notation $\delta^{(k)}(x)$ indicates that $\delta^{(k)}(x)$ is the distributional derivative of order k of $\delta(x)$, that is, $\delta^{(k)}(x) = (d^k / dx^k)\delta(x)$. The generalized function $\delta^{(k)}(x)$ are support at {0}, that is, (4) vanishes if the support of ϕ does not contain the point $x = 0$.

Another important formula is the orthogonality relation

$$\left\langle \delta^{(k)}(x), x^l \right\rangle = \begin{cases} 0 & \text{for } k \neq l, \\ (-1)^k k! & \text{for } k = l. \end{cases} \tag{5}$$

Thus the *k*th term in the sequences $\{x^k / k!\}$ and $\{(-1)^k \delta^{(k)}(x)\}$ are biorthogonal or dual to each other.

Let us now consider a series of the form

$$y(x) = \sum_{k=0}^{\infty} a_k \delta^{(k)}(x), \tag{6}$$

where all a_k are constants. Observe that the series diverges, in the distributional sense, unless the number of a_k that does not vanish is finite. In fact, if the series is convergent, then for each $\phi \in D(\mathbb{R})$, we have that

$$\left\langle y(x), \phi(x) \right\rangle = \left\langle \sum_{k=0}^{\infty} a_k \delta^{(k)}(x), \phi(x) \right\rangle = \sum_{k=0}^{\infty} (-1)^k a_k \phi^{(k)}(0).$$

According to a well-known theorem of Borel (3), given an arbitrary sequence of real or complex numbers $\{b_k\}$, then exists a smooth function ϕ that can be taken in $D(\mathbb{R})$ with $\phi^{(k)}(0) = b_k$, for $k = 0, 1, 2, \ldots.$ It follows that the series $\sum_{k=0}^{\infty} a_k b_k$ converges for every arbitrary sequence $\{b_k\}$, and thus there are exists N such that $a_k = 0$ for all $k > N$.

Unfortunately, convergence in spaces of ultra-distributions, or more generally, in spaces of hyper functions, is not much better. For instance, it follows from the general theory of hyper function (Kaneko, 1989) that, if

$$\overline{\lim_{k \to \infty}} \left(|a_k| \, k! \right)^{1/k} > 0, \tag{7}$$

then the series (6) does not define a hyper function.

Consequently, in the general case, series of Dirac delta functions cannot be considered as distributions, ultra distributions, or even as hyper functions. Naturally, however, any of the series can be considered as a functional in the space of polynomials. The term "dual Taylor series" can be used to identify them, since, in a sense, they are "dual" to the Taylor series $\sum_{k=0}^{\infty} b_k x^k$.

We can apply the usual vector space operations and other operations such as multiplication by polynomials and differentiation to dual Taylor series. In fact, we have

$$\frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k \delta^{(k)}(x)\right) = \sum_{k=0}^{\infty} a_{k-1}\delta^{(k)}(x), \tag{8}$$

$$x\sum_{k=0}^{\infty} a_k \delta^{(k)}(x) = -\sum_{k=0}^{\infty}(k+1)a_{k+1}\delta^{(k)}(x). \tag{9}$$

The multiplication $\phi(x)\sum_{k=0}^{\infty} a_k \delta^{(k)}(x)$ is well-defined if ϕ is a polynomial. However, if an infinite number of derivative values $\phi^{(k)}(0)$ does not vanish, then the product $\phi(x)\sum_{k=0}^{\infty} a_k \delta^{(k)}(x)$ may not make sense.

$$[m/2] = \begin{cases} \dfrac{m}{2}, & \text{if } m \ \textit{is even} \\[2ex] \dfrac{m-1}{2}, & \text{if } m \ \textit{is odd}. \end{cases} \tag{10}$$

Results and discussion

Theorem 3.1 *Suppose that* $w(x) = \sum_{k=0}^{\infty} a_k \delta^{(k)}(x)$ *is a formal distributional solution to nth-order differential equation*

$$x^2 y^{(n)}(x) + xy^{(n-1)}(x) + (b^2x^2 - p^2)y^{(n-2)}(x) = 0, \tag{11}$$

where $p \geq 0, n \geq 2, b > 0$ *and* x *is a real variable. Then (i) if* $n = 2$ *, then*

$$w_1(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k \prod_{j=1}^{k}(2j-1-p)\prod_{j=1}^{k}(2j-1+p)}{b^{2k}(2k)!}\delta^{(2k)}(x); \tag{12}$$

and

$$w_2(x) = a_1 \sum_{k=0}^{\infty} \frac{(-1)^k \prod_{j=1}^{k}(2j-p)\prod_{j=1}^{k}(2j+p)}{b^{2k}(2k+1)!}\delta^{(2k+1)}(x); \tag{13}$$

(ii) if $n = 3$ *, then*

$$w(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k \prod_{j=1}^{k}(2j-p)\prod_{j=1}^{k}(2j+p)}{b^{2k}(2k+1)!}\delta^{(2k)}(x); \tag{14}$$

(iii) if $n \geq 4$ *, then*

$$w(x) \equiv 0. \tag{15}$$

Proof. First we have to appeal to the basic concept of distribution theory, namely, use the test function $\phi(x) \in D(\mathbb{R})$. Accordingly, we have to examine the quantity

$$\left\langle x^2 y^{(n)} + xy^{(n-1)} + (b^2x^2 - p^2)y^{(n-2)}, \phi \right\rangle = \left\langle x^2 y^{(n)}, \phi \right\rangle + \left\langle xy^{(n-1)}, \phi \right\rangle + \left\langle (b^2x^2 - p^2)y^{(n-2)}, \phi \right\rangle. \quad (16)$$

Now

$$\left\langle x^2 y^{(n)}, \phi \right\rangle = \left\langle y^{(n)}, x^2\phi \right\rangle = (-1)^n \left\langle y, \left(x^2\phi\right)^{(n)} \right\rangle = (-1)^{(n)} \left\langle y, x^2\phi^{(n)} + 2nx\phi^{(n-1)} + n(n-1)\phi^{(n-2)} \right\rangle, \quad (17)$$

$$\left\langle xy^{(n-1)}, \phi \right\rangle = \left\langle y^{(n-1)}, x\phi \right\rangle = (-1)^{n-1} \left\langle y, \left(x\phi\right)^{(n-1)} \right\rangle = (-1)^{(n-1)} \left\langle y, x\phi^{(n-1)} + (n-1)\phi^{(n-2)} \right\rangle \quad (18)$$

and

$$\left\langle (b^2x^2 - p^2)y^{(n-2)}, \phi \right\rangle = \left\langle y^{(n-2)}, (b^2x^2 - p^2)\phi \right\rangle = (-1)^{(n-2)} \left\langle y, \left((b^2x^2 - p^2)\phi\right)^{(n-2)} \right\rangle$$
$$= (-1)^{(n-2)} \left\langle y, (b^2x^2 - p^2)\phi^{(n-2)} + 2(n-2)b^2x\phi^{(n-3)} + (n-2)(n-3)b^2\phi^{(n-4)} \right\rangle \quad (19)$$

Next, we substitute the series $y(x) = \sum_{k=0}^{\infty} a_k \delta^{(k)}(x)$ in the right-hand side of (17) to (19), we have

$$\left\langle xy^{(n-1)}, \phi \right\rangle = \sum_{k=0}^{\infty} a_k \left\langle -(k+n-1)\delta^{(k+n-2)}(x), \phi \right\rangle,$$
$$\left\langle xy^{(n-1)}, \phi \right\rangle = \sum_{k=0}^{\infty} a_k \left\langle -(k+n-1)\delta^{(k+n-2)}(x), \phi \right\rangle$$

and

$$\left\langle (b^2x^2 - p^2)y^{(n-2)}, \phi \right\rangle = \left\langle \sum_{k=0}^{\infty} a_k b^2 \left[k(k-1) + 2(n-2)k + (n-2)(n-3)\right] \delta^{(k+n-4)}(x) - \sum_{k=0}^{\infty} a_k p^2 \delta^{(k+n-4)}(x), \phi \right\rangle.$$

Substituting the values in (16), we obtain

$$\left\langle x^2 y^{(n)} + xy^{(n-1)} + (b^2x^2 - p^2)y^{(n-2)}, \phi \right\rangle$$
$$= \left\langle \sum_{k=0}^{\infty} \left[a_k (k+n-1-p)(k+n-1+p) + a_{k+2}(k+n)(k+n-1)\right] \delta^{(k+n-2)}(x) \right.$$
$$\left. a_0 b^2 (n-2)(n-3)\delta^{(n-4)}(x) + a_1 b^2 (n-2)(n-1)\delta^{(k+n-4)}(x), \phi \right\rangle. \quad (20)$$

From (20) it follows that, if $y(x)$ is a solution to (11), then for $k = 0, 1, 2, ...,$ we have the recurrence relation

$$a_k (k+n-1-p)(k+n-1+p) + a_{k+2} b^2 (k+n)(k+n-1) = 0. \quad (21)$$

In order to find the coefficients a_{k+2}, we consider the following three cases:

Case 1: If $n>3$, then $b^2(n-2)(n-3)\neq 0$ and $b^2(n-2)(n-1)\neq 0$, and thus

$$a_{k+2}=a_k\frac{(k+n-1-p)(k+n-1+p)}{b^2(k+n)(k+n-1)}. \quad (22)$$

But since $a_0=0$ and $a_1=0$, we find that $a_{k+2}=0$ for all $k\geq 0$. This therefore yields $y(x)=0$.

Case2: If $n=2$, then $a_0\neq 0$ and $a_1\neq 0$, we have the recurrence relation

$$a_{k+2}=-a_k\frac{(k+1-p)(k+1+p)}{b^2(k+2)(k+1)}. \quad (23)$$

Now, from the recurrence relation (23), we consider

$$a_2=-a_0\frac{(1-p)(1+p)}{b^2\cdot 1\cdot 2},$$

$$a_4=-a_2\frac{(3-p)(3+p)}{b^2\cdot 3\cdot 4}=a_0\frac{(1-p)(3-p)(1+p)(3+p)}{b^4\cdot 1\cdot 2\cdot 3\cdot 4},$$

$$a_6=-a_4\frac{(5-p)(5+p)}{b^2\cdot 5\cdot 6}=-a_0\frac{(1-p)(3-p)(5-p)(1+p)(3+p)(5+p)}{b^6\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},$$

$$\vdots$$

$$a_{2r}=(-1)^r a_0\frac{(1-p)(3-p)\cdots(2r-1-p)(1+p)(3+p)\cdots(2r-1+p)}{b^{2r}\cdot 1\cdot 2\cdot 3\cdots 2r}$$

$$=(-1)^r a_0\frac{\prod_{j=1}^{r}(2j-1-p)\prod_{j=1}^{r}(2j-1+p)}{b^{2r}\cdot(2r)!}.$$

Similarly,

$$a_3=-a_1\frac{(2-p)(2+p)}{b^2\cdot 2\cdot 3},$$

$$a_5=-a_3\frac{(4-p)(4+p)}{b^2\cdot 4\cdot 5}=a_1\frac{(2-p)(4-p)(2+p)(4+p)}{b^4\cdot 2\cdot 3\cdot 4\cdot 5},$$

$$a_7=-a_5\frac{(6-p)(6+p)}{b^2\cdot 6\cdot 7}=-a_1\frac{(2-p)(4-p)(6-p)(2+p)(4+p)(6+p)}{b^6\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7},$$

$$\vdots$$

$$a_{2r+1}=(-1)^r a_1\frac{(2-p)(4-p)\cdots(2r-p)(2+p)(4+p)\cdots(2r+p)}{b^{2r}\cdot 1\cdot 2\cdot 3\cdots(2r+1)}$$

$$=(-1)^r a_1\frac{\prod_{j=1}^{r}(2j-p)\prod_{j=1}^{r}(2j+p)}{b^{2r}\cdot(2r+1)!}.$$

Therefore, we have the solution

$$y_1(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k \prod_{j=1}^{r}(2j-1-p)\prod_{j=1}^{r}(2j-1+p)}{b^{2k}\cdot(2k)!}\delta^{(2k)}(x), \tag{24}$$

and

$$y_2(x) = a_1 \sum_{k=0}^{\infty} \frac{(-1)^k \prod_{j=1}^{r}(2j-p)\prod_{j=1}^{r}(2j+p)}{b^{2k}\cdot(2k+1)!}\delta^{(2k+1)}(x). \tag{25}$$

Case3: If $n = 3$, then $a_0 \neq 0$ and $a_1 = 0$, we have the recurrence relation

$$a_{k+2} = -a_r \frac{(k+2-p)(k+2+p)}{b^2(k+3)(k+2)}. \tag{26}$$

From there currency relation (26), we consider

$$a_2 = -a_0 \frac{(2-p)(2+p)}{b^2\cdot 3\cdot 2},$$

$$a_4 = -a_2 \frac{(4-p)(4+p)}{b^2\cdot 5\cdot 4} = a_0 \frac{(2-p)(4-p)(2+p)(4+p)}{b^4\cdot 5\cdot 4\cdot 3\cdot 2},$$

$$a_6 = -a_4 \frac{(6-p)(6+p)}{b^2\cdot 7\cdot 6} = -a_0 \frac{(2-p)(4-p)(6-p)(2+p)(4+p)(6+p)}{b^6\cdot 6\cdot 7\cdot 5\cdot 4\cdot 3\cdot 2},$$

$$\vdots$$

$$a_{2r} = (-1)^r a_0 \frac{(2-p)(4-p)\cdots(2r-p)(2+p)(4+p)\cdots(2r+p)}{b^{2r}\cdot 1\cdot 2\cdot 3\cdots(2r+1)}$$

$$= (-1)^r a_0 \frac{\prod_{j=1}^{r}(2j-p)\prod_{j=1}^{r}(2j+p)}{b^{2r}\cdot(2r+1)!}.$$

Thus, we obtain solutions is

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k \prod_{j=1}^{r}(2j-p)\prod_{j=1}^{r}(2j+p)}{b^{2k}\cdot(2k+1)!}\delta^{(2k)}(x). \tag{27}$$

This completes the proofs.

Corollary 3.1 *Let m be a non-negative integer. Then the distributional solution to the differential equation of*

$$x^2y''(x)+xy'(x)+(b^2x^2-(m+1)^2)y(x)=0 \quad (28)$$

is in the form

$$y(x)=C\sum_{k=0}^{[m/2]}\frac{b^{2k}(m-k)!}{4^k k!(m-2k)!}\delta^{(m-2k)}(x). \quad (29)$$

where C is any constant.

Corollary 3.2 *Let q be a non-negative integer. The n the distributional solution to the differential equation of*

$$x^2y'''(x)+xy'''(x)-(x^2+(q+1)^2)y'(x)=0 \quad (30)$$

is of the form

$$y(x)=C\sum_{k=0}^{\frac{q-1}{2}}\frac{b^{2k}(q-k)!}{4^k k!(q-2k)!}\delta^{(q-(2k+1))}(x). \quad (31)$$

where C is any constant.

Example 3.1 If $b=1,$ the equation (11) is the Bessel equation

1. For $p=1$, the equation (11) becomes

$$x^2y''+xy'+(x^2-1)y=0. \quad (32)$$

It follows from (12) that its solution is $y_0=a_0\delta(x).$

2. For $p=2$, the equation (11) becomes

$$x^2y''+xy'+(x^2-4)y=0. \quad (33)$$

It follows from (12) that its solution is $y_1(x)=a_0\delta'(x).$

3. For $p=3$, the equation (11) becomes

$$x^2y''+xy'+(x^2-9)y=0. \quad (34)$$

It follows from (12) that its solution is

$$\begin{aligned} y_2(x) &= C\sum_{k=0}^{1}\frac{(2-k)!}{4^k k!(2-2k)!}\delta^{(2-2k)}(x) \\ &= 4a_0\left(\frac{1}{4}\delta(x)+\delta''(x)\right) \\ &= a_0(\delta(x)+4\delta''(x)). \end{aligned}$$

4. For $p=4$, the equation (11) becomes

$$x^2y''+xy'+(x^2-16)y=0. \quad (35)$$

It follows from (12) that its solution is

$$\begin{aligned} y_3(x) &= C\sum_{k=0}^{1}\frac{(3-k)!}{4^k k!(3-2k)!}\delta^{(3-2k)}(x) \\ &= 2a_0\left(\frac{1}{2}\delta'(x)+\delta'''(x)\right) \\ &= a_1(\delta'(x)+2\delta'''(x)). \end{aligned}$$

5. For $p=5$, the equation (11) becomes

$$x^2y''+xy'+(x^2-25)y=0. \quad (36)$$

It follows from (12) that its solution is

$$y_4(x) = C\sum_{k=0}^{2}\frac{(4-k)!}{4^k k!(4-2k)!}\delta^{(4-2k)}(x)$$
$$= 16a_0\left(\frac{1}{16}\delta(x)+\frac{3}{4}\delta''(x)+\delta^{(4)}(x)\right)$$
$$= a_0(\delta(x)+12\delta''(x)+16\delta^{(4)}(x)).$$

6. For $p = 6$, the equation (11) becomes

$$x^2y''+xy'+(x^2-36)y=0. \quad (37)$$

It follows from (12) that its solution is

$$y_5(x) = C\sum_{k=0}^{2}\frac{(5-k)!}{4^k k!(5-2k)!}\delta^{(5-2k)}(x)$$
$$=\frac{16}{3}a_1\left(\frac{3}{16}\delta'(x)+\delta'''(x)+\delta^{(5)}(x)\right)$$
$$=a_1\left(\delta'(x)+\frac{16}{3}\delta'''(x)+\frac{16}{3}\delta^{(5)}(x)\right).$$

7. For $p = 7$, the equation (11) becomes

$$x^2y''+xy'+(x^2-49)y=0. \quad (38)$$

It follows from (12) that its solution is

$$y_6(x) = C\sum_{k=0}^{3}\frac{(6-k)!}{4^k k!(6-2k)!}\delta^{(6-2k)}(x)$$
$$=64a_0\left(\frac{1}{64}\delta(x)+\frac{3}{8}\delta''(x)+\frac{5}{4}\delta^{(4)}(x)+\delta^{(6)}(x)\right)$$
$$=a_0(\delta(x)+24\delta''(x)+80\delta^{(4)}(x)+64\delta^{(6)}(x)).$$

Conclusion

In this research, we get the main result

Theorem: *Suppose that* $w(x)=\sum_{k=0}^{\infty}a_k\delta^{(k)}(x)$ *is a formal distributional solution to nth-order differential equation*

$$x^2y^{(n)}(x)+xy^{(n-1)}(x)+(b^2x^2-p^2)y^{(n-2)}(x)=0,$$

where $p\geq 0, n\geq 2, b>0$ *and* x *is a real variable. Then (i) if* $n=2$ *, then*

$$w_1(x)=a_0\sum_{k=0}^{\infty}\frac{(-1)^k\prod_{j=1}^{k}(2j-1-p)\prod_{j=1}^{k}(2j-1+p)}{b^{2k}(2k)!}\delta^{(2k)}(x);$$

and

$$w_2(x)=a_1\sum_{k=0}^{\infty}\frac{(-1)^k\prod_{j=1}^{k}(2j-p)\prod_{j=1}^{k}(2j+p)}{b^{2k}(2k+1)!}\delta^{(2k+1)}(x);$$

(ii) if $n=3$ *, then*

$$w(x)=a_0\sum_{k=0}^{\infty}\frac{(-1)^k\prod_{j=1}^{k}(2j-p)\prod_{j=1}^{k}(2j+p)}{b^{2k}(2k+1)!}\delta^{(2k)}(x);$$

(iii) if $n\geq 4$ *, then*

$$w(x)\equiv 0.$$

Corollary 1 *Let* m *be a non-negative integer. Then the distributional solution to the differential equation of*

$$x^2y''(x)+xy'(x)+(b^2x^2-(m+1)^2)y(x)=0$$

is in the form

$$y(x)=C\sum_{k=0}^{[m/2]}\frac{b^{2k}(m-k)!}{4^k k!(m-2k)!}\delta^{(m-2k)}(x).$$

where C is any constant.

Corollary 2 *Let* q *m be a non-negative integer. The n the distributional solution to the differential equation of*

$$x^2y'''(x)+xy''(x)-(x^2+(q+1)^2)y'(x)=0$$

is of the form

$$y(x)=C\sum_{k=0}^{\frac{q-1}{2}}\frac{b^{2k}(q-k)!}{4^k k!(q-2k)!}\delta^{(q-(2k+1))}(x).$$

where C is any constant.

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References

Cooke, K.L. and J. Wiener. 1984. Distributional and analytics solutions of functional differential equations. *J. Math. Anal. Appl. 98*: 111-129.

Hernandez-Urena, L.G. and R. Estrada. 1995. Solutions of ordinary differential equations by series of delta functions. *J. Math. Anal. Appl. 191*: 40-55.

Kaneko, A. 1989. Introduction to hyperfunctions. Kluwer Press, Boston.

Kanwal, R.P. 1983. Generalized functions: Theory and technique. Academic Press, New York.

Krall, A.M. 1981. Orthogonal polynomials satisfying fourth order differential equation. *Proc. Roy. Soc. Edinburgh Sect. 87(A)*: 271-288.

Littlejohn, L.L. 1984. On the classification of differential equations having orthogonal polynomial solutions. *Ann. Mat. Pura Appl. 138*: 35-53.

Littlejohn, L.L. and R.P. Kanwal. 1987. Distributional solutions of the hypergeomatric differential equation. *J. Math. Anal. Appl. 12*: 325-345.

Morton, R.D. and A.M. Krall. 1978. Distributional weight functions for orthogonal polynomials. *SIAM J. Math. Anal. 9*: 604-626.

Schwartz, L. 1957. Theorie des Distributions, vol. 1, Actualite's Scientifiques et Industrial. Hermann, Paris.

Schwartz, L. 1959. Theorie des Distributions, vol. 2, Actualite's Scientifiques et Industrial. Hermann, Paris.

Shah, S.M. and J. Wiener. 1983. Distributional and entire solutions of ordinary differential and functional differential equations. *Int. J. Math. Math. Sci. 6(2)*: 243-270.

Wiener, J. 1982. Generalized-function solutions of the differential and functional differential equations. *J. Math. Anal. Appl. 88*: 170-182.

Wiener, J. and K.L. Cooke. 1990. Coexistence of analytic and distributional solutions for linear differential equations I. *J. Math. Anal. Appl. 148*: 390-421.

Wiener, J., K.L. Cooke and S.M. Shah. 1991. Coexistence of analytic and distributional solutions for linear differential equations II. *J. Math. Anal. Appl. 159*: 271-289.