ระเบียบวิธีการทำซ้ำอันดับสามรูปแบบใหม่สำหรับการแก้สมการไม่เชิงเส้น

A new third-order Iterative method for solving nonlinear equations

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บทคัดย่อ

ในงานวิจัยนี้ได้นำเสนอระเบียบวิธีการทำซ้ำรูปแบบใหม่สำหรับการหาผลเฉลยของสมการไม่เชิงเส้น บนพื้นฐานการประมาณค่าด้วยสมการกำลังสอง $x^2+ax+by+c=0$ เมื่อ a,b,c เป็นค่าคงที่ เราได้พิสูจน์ระเบียบวิธีการทำซ้ำรูปแบบใหม่นี้มีอันดับการลู่เข้าอันดับสามและได้นำเสนอตัวอย่างเชิงตัวเลขของระเบียบวิธีการทำซ้ำรูปแบบใหม่นี้กับระเบียบวิธีการทำซ้ำที่มีอันดับการลู่เข้าอันดับสามรูปแบบอื่น พบว่าระเบียบวิธีการทำซ้ำรูปแบบใหม่สามารถเป็นอีกหนึ่งตัวเลือกในการแก้สมการไม่เชิงเส้น

คำสำคัญ : สมการไม่เชิงเส้น การลู่เข้า วิธีการทำซ้ำ

Abstract

In this paper, we present a new iterative method for finding a root of nonlinear equations. The new method is based on approximation by the quadratic equation $x^2+ax+by+c=0$ where a,b,c are constants. We prove that the new method has cubic convergence. Several numerical examples are presented and we compared the numerical solutions of new method with other cubic convergence methods. We found out the new method can be an alternative to solve nonlinear equations.

Keywords : Nonlinear equations; Convergence; Iterative methods

1. Introduction

$$f(x)=0 \qquad (1)$$

is an important problem in sciences and engineering. We know that the way to obtain an approximate solution of (1) is using numerical techniques based on iteration procedures. Recently, there are many iterative numerical methods to find the root of (1). These methods were constructed by different techniques. One of the techniques is geometric construction. Let r be the root and x_0 be the initial approximation to the root of (1). Mamta et.al. [1] considered straight line which has slope $mf(x_0)$ and pass through the point $(x_0,0)$, in the form

$$y=mf(x_0)(x-x_0) \qquad (2)$$

where $m\in\square$.

Let $x_1=x_0+h$ be an approximation to the root where $|h|<1$. If the straight line (2) coincides the graph of $y=f(x)$ at the point $(x_1,f(x_0+h))$, then we get

$$f(x_1)=mf(x_0)(x_1-x_0)$$

or $$f(x_0+h)=mhf(x_0). \qquad (3)$$

Expanding the value of $f(x_0+h)$ by Taylor's expansion around x_0, we get

$$f(x_0+h)=f(x_0)+hf'(x_0)+O(h^2).$$

Since $O(h^2)\to 0$, we can ignore this term so we have

$$f(x_0+h)\approx f(x_0)+hf'(x_0). \quad (4)$$

Substituting the value of (4) into (3), we have

$$f(x_0)+hf'(x_0)=mhf(x_0)$$

or

$$h(f'(x_0)-mf(x_0))=-f(x_0).$$

Then

$$h=-\frac{f(x_0)}{f'(x_0)-mf(x_0)}. \quad (5)$$

Since $x_1=x_0+h$,

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)-mf(x_0)} \quad (6)$$

Therefore we get the iterative method as follows

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)-mf(x_n)}\quad,\quad n=0,1,2,... \quad (7)$$

This method has a quadratic convergence.

If we fit the function $f(x)$ with parabola

$$y(x)=x^2+ax+by+c \quad (8)$$

at a point $x=x_n$, Equation (8) becomes:

$$y(x_n)=x_n^{\ 2}+ax_n+by+c. \quad (9)$$

Imposing the tangency conditions,

$$y(x_n)=f(x_n),\ y'(x_n)=f'(x_n),\ y''(x_n)=f''(x_n),$$

we get

$$y(x)-f(x_n)=\frac{f''(x_n)(x-x_n)^2}{2}+(x-x_n)f'(x_n) \quad (10)$$

Assume x_{n+1} is the root then $y(x_{n+1})=0$ and replace x by x_{n+1} in (10), we get

$$(x_{n+1}-x_n)^2\frac{f''(x_n)}{2}+(x_{n+1}-x_n)f'(x_n)+f(x_n)=0. \quad (11)$$

Solving equation (11) for the root x_{n+1}, we get the irrational Halley's method [2] as follows,

$$x_{n+1}=x_n-\frac{2}{1+\sqrt{1-2L(x_n)}}\frac{f(x_n)}{f'(x_n)}\quad,\quad n\ge 0 \quad (12)$$

where

$$L(x_n)=\frac{f(x_n)f''(x_n)}{(f'(x_n))^2}.$$

This method has a third-order convergence.

We know that the computation of the square root is not convenient for use. Therefore in this paper will present a new iterative method based on approximation by the quadratic equation $x^2+ax+by+c=0$ where a,b,c are constants and we will describe in the next section.

2. The method

From (11), we have

$$x_{n+1}=x_n-\frac{f(x_n)}{\left[\frac{f''(x_n)(x_{n+1}-x_n)}{2}+f'(x_n)\right]}. \quad (13)$$

Since (13) is an implicit equation, to overcome this drawback we replace x_{n+1} in the right of equation by y_n where

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)+f(x_n)}. \quad (14)$$

Substituting the value of y_n into (13), we get

$$x_{n+1}=x_n-\frac{f(x_n)}{\left[-\frac{f(x_n)f''(x_n)}{2(f'(x_n)+f(x_n))}\right]+f'(x_n)}.$$

Therefore, we can conclude that

$$x_{n+1}=x_n-\frac{2[f(x_n)f'(x_n)+(f(x_n))^2]}{2f(x_n)f'(x_n)-f''(x_n)f(x_n)+2(f'(x_n))^2},$$

then we obtain the new iterative method as follows;

Algorithm 1: For a given x_0, compute a root x_{n+1} by the iterative method

$$x_{n+1}=x_n-\frac{2[f(x_n)f'(x_n)+(f(x_n))^2]}{2f(x_n)f'(x_n)-f''(x_n)f(x_n)+2(f'(x_n))^2},\ n\geq 0 \quad (15)$$

Note that the value of y_n in (14) is obtained from Mamta's method in the case when $m=-1$.

3. Convergence

Theorem 1. Let $f:D\rightarrow$ □ for an open interval D. Assume that f is sufficiently differentiable in the interval D. If $f(x)$ has a simple root at $\alpha\in D$ and x_0 is sufficiently close to α, then the method defined by (15) converges quadratic to α and satisfies the error equation

$$e_{n+1}=(c_3-c_2-c_2^2)e_n^3+O(e_n^4).$$

\(16)

Proof. Let $x_n=e_n+\alpha$. Using Taylor expansion around $x=\alpha$ and $f(\alpha)=0$, we have

$$f(x_n)=f'(\alpha)[e_n+c_2e_n^2+c_3e_n^3+O(e_n^4)].(17)$$

Furthermore, we have

$$f'(x_n)=f'(\alpha)[1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+O(e_n^4)] \quad (18)$$

and

$$f''(x_n)=f'(\alpha)[2c_2+6c_3e_n+12c_4e_n^2+20c_5e_n^3+O(e_n^4)] \quad (19)$$

where

$$c_k=\left(\frac{1}{k!}\right)\frac{f^{(k)}(\alpha)}{f'(\alpha)},k=2,3,4,\ldots$$

From (17) - (19), we have

$$(f(x_n))^2=(f'(\alpha))^2[e_n^2+2c_2e_n^3+(2c_3+c_2^2)e_n^4+O(e_n^5)] \quad (20)$$

$$(f'(x_n))^2=(f'(\alpha))^2[1+4c_2e_n+(4c_2^2+6c_3)e_n^2+(8c_4+12c_2c_3)e_n^3+(16c_2c_4+9c_3^2)e_n^4+O(e_n^5)] \quad (21)$$

$$f(x_n)f'(x_n)=(f'(\alpha))^2[e_n+3c_2e_n^2+(4c_3+2c_2^2)e_n^3+(5c_4+5c_2c_3)e_n^4+O(e_n^5)] \quad (22)$$

$$f''(x_n)f(x_n)=(f'(\alpha))^2[2c_2e_n+(2c_2^2+6c_3)e_n^2+(8c_2c_3+12c_4)e_n^3+(20c_5+14c_2c_4+6c_3^2)e_n^4+O(e_n^5)]. \quad (23)$$

Substituting the values of (20)-(23) into $\frac{2[f(x_n)f'(x_n)+(f(x_n))^2]}{2f(x_n)f'(x_n)-f''(x_n)f(x_n)+2(f'(x_n))^2}$, we get

$$\frac{2[f(x_n)f'(x_n)+(f(x_n))^2]}{2f(x_n)f'(x_n)-f''(x_n)f(x_n)+2(f'(x_n))^2}=[e_n+(c_3-c_2-c_2^2)e_n^3+O(e_n^4)].$$

From

$$x_{n+1}=x_n-\frac{2[f(x_n)f'(x_n)+(f(x_n))^2]}{2f(x_n)f'(x_n)-f''(x_n)f(x_n)+2(f'(x_n))^2}$$

and $e_{n+1}=x_{n+1}+\alpha$

we get

$$e_{n+1}+\alpha=e_n+\alpha-[e_n+(c_3-c_2-c_2^2)e_n^3+O(e_n^4)]$$

or $e_{n+1}=(c_3-c_2-c_2^2)e_n^3+O(e_n^4)$. Therefore this method has cubic convergence.

4. Numerical Examples

To show the performance and efficiency of the new method, we compare our method (new) to the following methods:

4.1 Iterational Halley's method (IH, [2]) defined by

$$x_{n+1}=x_n-\frac{2}{1+\sqrt{1-2L(x_n)}}\frac{f(x_n)}{f'(x_n)},\quad n\geq 0$$

where

$$L(x_n)=\frac{f(x_n)f''(x_n)}{(f'(x_n))^2}.$$

$$x_{n+1}=x_n-\frac{f(x_n)}{(\ \ldots\ )},\quad n\geq 0.$$

4.2 Householder's method (HH, [3]) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2(f'(x_n))^3} \quad , \quad n \ge 0$$

.

4.3 The method of Weerakoon and Fernando (WF, [4]) defined by

All of these methods have cubic convergence. use $\varepsilon = 10^{-15}$, then the stopping criteria $|f(x_{n+1})| < \varepsilon$. Results are presented in Table 1.

Table1: Results of the problem $f(x) = 0$

$f(x)$	**Initial guess** x_0	**root**	**Number of iterations**			
			new	**WF**	**IH**	**HH**
$x^2 - e^x - 3x + 2$	1.5	0.257530285439860	3	3	5	4
$e^{x^2+7x-30} - 1$	3.5	3.000000000000000	7	8	15	8
$xe^{x^2} - \sin^2 x + 3\cos x + 5$	-2	-1.207647827130919	4	6	10	8
$e^x + x - 20$	3.5	2.842438953784450	4	4	4	4
$e^{-x} + \cos x$	2	1.746139530408010	3	3	3	3
$\sin x - \frac{x}{2}$	2	1.895494267033980	3	3	3	3
$(x-1)^3 - 1$	2.5	2.000000000000000	4	4	4	4
$x^3 - 2x - 5$	2	2.094551481542330	3	3	3	3
$x^3 + \log(x)$	3	0.6032354400268530	5	5	Div	5

5. Conclusion

In this paper, we present an iterative method for solving nonlinear equation. This method is based on an approximation by the quadratic equation and includes a well-known method as particular ones. The error equation shows that new method is cubically convergent. The results from Table 1 show that the new method can be considered as an alternative method for solving nonlinear equations.

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