ลักษณะของแกมมากึ่งไฮเพอร์กรุปอันดับโดยใช้สมบัติของ (m,n)-ควอซีแกมมาไฮเพอร์ไอดีลอันดับ
Characterizations of Ordered Γ-semihypergroups by the Properties of Their Ordered (m,n)-Quasi-Γ-hyperideals

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บทคัดย่อ

ในบทความวิจัยนี้ เราได้ขยายแนวความคิดจากบทความวิจัยของ S.Thongrak และ A.Lampan (2018) ซึ่งเราได้แนะนำแนวคิดและคุณสมบัติของ m-แกมมาไฮเพอร์ไอดีลอันดับซ้าย, n-แกมมาไฮเพอร์ไอดีลอันดับขวาและ (m,n)-ควอซีแกมมาไฮเพอร์ไอดีลอันดับ ตามลำดับ และสุดท้ายเราได้แนะนำแนวคิดของ (m,n)-คุณสมบัติแกมมาส่วนร่วมอันดับ ในแกมมากึ่งไฮเพอร์กรุปอันดับ และพิสูจน์ว่าทุก ๆ (m,n)-ควอซีแกมมาไฮเพอร์ไอดีลอันดับ ในแกมมากึ่งไฮเพอร์กรุปอันดับปรกติ จะมี (m,n)-คุณสมบัติแกมมาส่วนร่วมอันดับ

คำสำคัญ: แกมมากึ่งไฮเพอร์กรุปอันดับ, (m,n)-ควอซีแกมมาไฮเพอร์ไอดีลอันดับ, (m,n) แกมมาสมบัติอินเตอร์เซกชัน, m-แกมมาไฮเพอร์ไอดีลอันดับซ้าย, n-แกมมาไฮเพอร์ไอดีลอันดับขวา

Abstract

This research was expanded from the research article of S.Thongrak and A.Lampan (2018). The concept and features of ordered m -left- Γ-hyperideals, ordered n -right-Γ-hyperideals were discussed and ordered (m,n)-quasi- Γ--hyperideals respectivety ,and the concept of ordered (m,n)- Γ-- intersection property in ordered - Γ semihypergroups and proved that every ordered (m,n)- quasi-Γ-hyperideals in regular ordered - semihypergroups had the ordered (m,n)-Γ-intersection property.

Keywords: Ordered Γ-semihypergroups, Ordered (m,n)-quasi-Γ-hyperideals, Ordered (m,n)-Γ-intersection property, Ordered m-left-Γ-hyperideals, Ordered n-right-Γ-hyperideals.

1. Introduction and Preliminaries

In 1986, M.K. Sen and N.K.Saha (Sen and Saha h 1986, pp. 180-186). define the notion of Γ-semi-group and that is the po-Γ-semigroup that was introduced by Y.I. Kwon and S.K. Lee (Kwon and lee, 1996, pp. 679-685). The notion of a quasi-ideals in semigroup was first invented by O. Steinfeld (Steinfeld, 1956, pp. 262-275). Thereafter, quasi-ideals have been studied in different algebraic structures in (Abbasi and Basar, 2013, pp. 1-7). N. Kehayopula, S. Lajos and G. Lepouras (Kehayopula, Lajos and Lepouras, 1997, pp. 75-81) defined and studied an ordered quasi-ideal in ordered semigroups. Recently, S. Thongrak and A. Lampan in (Thongrak and Lampan , 2018, pp. 299-306) gave the characterizations of ordered semigroups andinvestigate the an ordered (m,n) -quasi-ideals in ordered semigroups.

Hyperstructures theory was introduced in 1934, F. Marty (Marty, 1934, pp. 45-49). defined hypergroups, began to analyze their properties and applied them to groups. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science and they are studied in many countries of the world. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set.

In continuation of the study, we characterize of ordered Γ-semihypergroups and investigate the an ordered (m,n)-quasi-Γ-hyperideals which extends the results in ordered Γ-semihypergroups.

In this section, we recall some necessary definitions, notations and properties of some algebraic structures.

Definition 1.1. (Marty, 1934) Let H be a non-empty set. Then the map $\circ : H \times H \to P^*(H)$ is called a hyperoperation, where $P^*(H)$ is the family of non-empty subsets of H. The couple $(H,\circ)$ is called a hypergroupoid.

In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we define $A \circ B = \bigcup_{a\in A, b\in B} a \circ b, = x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$.

A hypergroupoid $(H,\circ)$ is called a semihypergroup if for every $x,y,z \in H$, we have

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

Definition 1.2. (Marty, 1934) An algebraic hyperstructure $(H,\circ,\leq)$ is called an ordered semihypergroup if $(H,\circ)$ is a semihypergroup and $(H,\leq)$ is a partially set such that the compatible condition hold as follows:

$$x \leq y \Rightarrow a \circ x \leq a \circ y \text{ and } x,y,a \in H,$$

where, if A and B are non-empty subsets of H, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

Definition 1.3. (Davvaz, Dehkordi & Heidari, 2010) Let H and Γ be two non-empty sets. H is called a Γ-semihypergroup if every $\gamma \in \Gamma$ is a hyperoperation on H, $x\gamma y \subseteq H$ for every $x,y \in H$, and for every $\alpha,\beta \in \Gamma$ and $x,y,z \in H$ we have

$$x\alpha(y\beta z) = (x\alpha y)\beta z.$$

Let A and B be two non-empty subsets of H and $\gamma \in \Gamma$. We define

$$A\gamma B = \bigcup_{a\in A, b\in B} a\gamma b \text{ and } A\Gamma B = \bigcup_{a\in A, b\in B} A\gamma B$$

In the following, we present the definition of an ordered Γ-semihypergroup and give some examples.

Definition 1.4. (Davvaz & Omidi, 2017) Let H and Γ be two non-empty sets. H is called anordered Γ-semihypergroup if H is a Γ-semihypergroup and $(H,\leq)$ is a partially ordered set such that the compatible condition hold as follows:

$$x \leq y \Rightarrow a\gamma x \leq a\gamma y \text{ and } x\gamma a \leq y\gamma a$$

for all $x,y \in H, \gamma \in \Gamma$, where, if A and B are non-empty subsets of H, then we say that $A \leq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$.

In the following, we denote an ordered Γ-semihypergroup $(H,\Gamma,\leq)$ by H unless otherwise specified.

A non-empty subset A of an ordered Γ-semihypergroup H is called a sub Γ-semihypergroup of H if $A\Gamma A \subseteq A$.

Definition 1.5. (Tang, Davvaz, Xie & Yaqoob, 2017) Let H be an ordered Γ-semihypergroup. A non-empty subset I of H is called a left (resp. right) Γ-hyperideal of H if

(i) $(H\Gamma I] \subseteq I$ (resp. $(I\Gamma H] \subseteq I$); and

(ii) If $a \in I$ and $b \in H$ such that $b \leq a$, then $b \in I$. Equivalent $(I] \subseteq I$.

If I is both a left Γ-hyperideal and a right Γ-hyperideal of H, then it is called Γ-hyperideal of H.

Definition 1.6. (Kondo & Lekkoksung, 2013) A non-empty subset Q of an ordered Γ-semihypergroup H is called a quasi-Γ-hyperideal of H if the following conditions hold:

(i) $(Q\Gamma H] \cap (H\Gamma Q] \subseteq Q$;

(ii) When $x \in Q$ and $y \in H$ such that $y \leq x$, implies $y \in Q$. Equivalent $(Q] \subseteq Q$.

Definition 1.7. Let H be an ordered Γ-semihypergroup. A sub-Γ-semihypergroup B of H is called an ordered m-left (resp. ordered n-right)-Γ-hyperideal of H if

(i) $H^m\Gamma B \subseteq B$, (resp. $B\Gamma H^n \subseteq B$),

(ii) for $x \in B$ and $y \in H$ such that $y \leq x$, implies $y \in B$. Equivalent $(B] \subseteq B$.

Definition 1.8. Assume that Q is a sub-Γ-semihypergroup of an orered-Γ-semihypergroup H. Then Q is called an ordered (m,n)-quasi-Γ-hyperideal of H if

(i) $(Q\Gamma H^n] \cap (H^m\Gamma Q] \subseteq Q$,

(ii) for $x \in Q$ and $y \in H$ such that $y \leq x$, implies $y \in Q$. Equivalent $(Q] \subseteq Q$.

Example 1. (Omidi & Davvaz, 2017) Let $H=\{a,b,c,d\}$ and $\Gamma=\{\gamma,\beta\}$ be the set of binary hyperopera-tions defined as follows:

γ	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	b	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

β	a	b	c	d
a	a	$\{a,b\}$	$\{c,d\}$	d
b	$\{a,b\}$	b	$\{c,d\}$	d
c	$\{c,d\}$	$\{c,d\}$	c	d
d	d	d	d	d

and $\leq:=\{(a,a),(a,b),(b,b),(c,b),(c,c),(c,d),(d,b),(d,d)\}$.

Then H is an ordered Γ-semihypergroup. Let $A=\{c,d\}$, we have that $H^1\Gamma A=\{c,d\}=A$ and $A\Gamma H^2=\{c,d\}=A$, also for every $c,d\in A$, there exists $c,d\in H$ such that $c\leq c, c\leq d, d\leq d$. implies that $(A]=A$. Thus A is an ordered 1-left-Γ-hyperideal and A is an ordered 2-right-Γ-hyperideal of H. Let $A=\{c,d\}$, we have that $(H^1\Gamma A]\cap(A\Gamma H^2]$ $=\{c,d\}\cap\{c,d\}=\{c,d\}=A$, also $(A]=A$. Hence A is an ordered (1,2)-quasi-Γ-hyperidealof H.

Example 2. (Omidi & Davvaz, 2017) Let $H=\{a,b,c,d,e\}$ and $\Gamma=\{\gamma,\beta\}$ be the sets of binary hyperoperations defined follows:

γ	a	b	c	d	e
a	$\{a,b\}$	$\{b,c\}$	c	$\{d,e\}$	e
b	$\{b,c\}$	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

β	a	b	c	d	e
a	$\{b,c\}$	c	c	$\{d,e\}$	e
b	c	c	c	$\{d,e\}$	e
c	c	c	c	$\{d,e\}$	e
d	$\{d,e\}$	$\{d,e\}$	$\{d,e\}$	d	e
e	e	e	e	e	e

and $\leq:=\{(a,a),(b,b),(c,c),(d,d),(e,e),(c,d),(e,c),(e,d)\}$.

Then H is an ordered Γ-semihypergroup. Let $A=\{d,e\}$, we have that $H^2\Gamma A=\{d,e\}=A$ and $A\Gamma H^3=\{d,e\}=A$, also for every $d,e\in A$, there exists $d,e\in H$ such that $d\leq d, d\leq e, e\leq d, e\leq e$ implies that $(A]=A$. Thus A is an ordered 2-left-Γ-hyperideal and A is an ordered 3-right-Γ-hyperideal of H. Let $A=\{d,e\}$, we have that $(H^2\Gamma A]\cap(A\Gamma H^3]$ $=\{d,e\}\cap\{d,e\}=\{d,e\}=A$, also $(A]=A$. Hence A is an ordered (2,3)-quasi-Γ-hyperideal of H.

Lemma 1.9. (Omidi & Davvaz, 2017) Let K be a non-empty subset of an ordered Γ-semihypergroup H we define $(K]:=\{x\in H \mid x\leq k$ for some $k\in K\}$. For $K=\{k\}$, we write $(k]$ instead of $(\{k\}]$. If A and B are non-empty subsets of H, then we have

(1) $A\subseteq(A]$;
(2) $((A]]=(A]$;
(3) If $A\subseteq B$, then $(A]\subseteq(B]$;
(4) $(A]\Gamma(B]\subseteq(A\Gamma B]$;
(5) $((A]\Gamma(B]]=(A\Gamma B]$;
(6) $(A\cap B]\subseteq(A]\cap(B]$;
(7) $(A\cup B]=(A]\cup(B]$.

Lemma 1.10. Let H be an ordered Γ-semihyper-group and $\{A_i \mid i\in I\}$ a non-empty family of sub-Γ-semihypergroup of H. Then $\bigcap_{i\in I}A_i=\varnothing$ or $\bigcap_{i\in I}A_i$ is a sub-Γ-semihypergroup of H.

Proof. Let $\{A_i \mid i\in I\}$ a non-empty family of sub-Γ-semihypergroup of H. Suppose that $\bigcap_{i\in I}A_i\neq\varnothing$. Let $a,b\in\bigcap_{i\in I}A_i\neq\varnothing$, we have $a,b\in A_i$ for all $i\in I$. since A_i is a sub-Γ-semihypergroup of H, $a\gamma b\subseteq A_i$ for all $i\in I$ and $\gamma\in\Gamma$. Hence $a\gamma b\subseteq\bigcap_{i\in I}A_i$.

Therefore $\bigcap_{i\in I}A_i$ is a sub-Γ-semihypergroup of H.

Lemma 1.11. Let H be an ordered Γ-semihyper-group and A a sub-Γ-semihypergroup of H. Then $A^n\subseteq A$ for all positive integer n.

Proposition 1.12. Let H be an ordered Γ-semi-hypergroup, Q an ordered (m,n)-quasi-Γ-hyper-ideal of H and A a sub-Γ-semihypergroup of H such that $(A]=A$. Then $A\cap Q=\varnothing$ or $A\cap Q$ is an ordered (m,n)-quasi-Γ-hyperideal of A.

Proof. Assume that $A\cap Q\neq\varnothing$. Since Q and A are sub-Γ-semihypergroup of H, we have $A\cap Q$ is a sub-Γ-semihypergroup of H. Since $A\cap Q\subseteq A$, we have $A\cap Q$ is a sub-Γ-semihypergroup of A. Thus

$$\begin{aligned}(A^m\Gamma(A\cap Q)]\cap((A\cap Q)\Gamma A^n] &\subseteq (A^m\Gamma Q]\cap(Q\Gamma A^n]\\ &\subseteq (H^m\Gamma Q]\cap(Q\Gamma H^n]\\ &\subseteq Q,\end{aligned}$$

and

$$\begin{aligned}(A^m\Gamma(A\cap Q)]\cap((A\cap Q)\Gamma A^n] &\subseteq (A^m\Gamma A]\cap(A\Gamma A^n]\\ &\subseteq (A]\cap(A]\\ &= A\cap A\\ &= A.\end{aligned}$$

We have

$$\begin{aligned}(A\cap Q] &\subseteq (A]\cap(Q]\\ &\subseteq A\cap Q.\end{aligned}$$

Hence, $(A\cap Q]=A\cap Q$. Therefore $A\cap Q$ is an ordered (m,n)-quasi-Γ-hyperideal of A.

Proposition 1.13. Let H be an ordered Γ-semihypergroup and $\{Q_i \mid i\in I\}$ a non-empty famity of ordered (m,n)-quasi-Γ-hyperideal of H. Then $\bigcap_{i\in I}Q_i=\varnothing$ or $\bigcap_{i\in I}Q_i$ is an ordered (m,n)-quasi-Γ-hyperideal of H.

Proof. Assume that $\bigcap_{i\in I}Q_i\neq\varnothing$. By Lemma 1.7, we have $\bigcap_{i\in I}Q_i$ is a sub-Γ-semihyper-grou of H. For all $i\in I$, we have $(H^m\Gamma(\bigcap_{i\in I}Q_i)]\cap((\bigcap_{i\in I}Q_i)\Gamma H^n]$ $\subseteq (H^m\Gamma Q_i]\cap(Q_i\Gamma H^n]\subseteq Q_i$.

Thus $(H^m\Gamma(\bigcap_{i\in I}Q_i)]\cap((\bigcap_{i\in I}Q_i)\Gamma H^m]\subseteq\bigcap_{i\in I}Q_i$ and $(\bigcap_{i\in I}Q_i]\subseteq\bigcap_{i\in I}(Q_i]=\bigcap_{i\in I}Q_i$. Therefore, $\bigcap_{i\in I}Q_i$ is an ordered (m,n)-quasi-Γ-hyperideal of H.

2. Main results

2.1. Ordered (m,n)-quasi-Γ-hyperideal and Ordered (m,n)-Γ-intersection Property

In this part, we characterize ordered m-left-Γ-hyperideals and ordered n-right-Γ-hyperideals in ordered Γ-semihypergroups and investigate the ordered (m,n)-Γ-intersection property of ordered (m,n)-quasi-Γ-hyperideals in ordered Γ-semihypergroups.

Theorem 2.1.1. Let H be an ordered Γ-semihypergroup. Then the following statements hold.

(i) If $\{A_i \mid i\in I\}$ is a non-empty family of ordered m-left-Γ-hyperideal of H, then $\bigcap_{i\in I}A_i=\varnothing$ or $\bigcap_{i\in I}A_i$ is an ordered m-left-Γ-hyperideal of H.

(ii) If $\{B_i \mid i\in I\}$ is a non-empty family of ordered n-right-Γ-hyperideal of H, then $\bigcap_{i\in I}B_i=\varnothing$ or $\bigcap_{i\in I}B_i$ is an ordered n-right-Γ-hyperideal of H.

Proof. (i) Assume that $\{A_i \mid i\in I\}$ is a non-empty family of ordered m-left-Γ-hyperideal of H and let $\bigcap_{i\in I}A_i\neq\varnothing$. By Lemma 1.7, we have $\bigcap_{i\in I}A_i$ is a sub-Γ-semihypergroup of H. For all $i\in I$, we have $H^m\Gamma(\bigcap_{i\in I}A_i)\subseteq H^m\Gamma A_i\subseteq A_i$. Thus $H^m\Gamma(\bigcap_{i\in I}A_i)\subseteq\bigcap_{i\in I}A_i$ and $(\bigcap_{i\in I}A_i]\subseteq\bigcap_{i\in I}(A_i]=\bigcap_{i\in I}A_i$. Therefore, $\bigcap_{i\in I}A_i$ is an ordered m-left-Γ-hyperideal of H.

(ii) Assume that $\{B_i \mid i\in I\}$ is a non-empty family of ordered n-right-Γ-hyperideal of H and let $\bigcap_{i\in I}B_i\neq\varnothing$. By Lemma 1.7, we have $\bigcap_{i\in I}B_i$ is a sub-Γ-semihypergroup of H. For all $i\in I$, we have $(\bigcap_{i\in I}B_i)\Gamma H^n\subseteq B_i\Gamma H^n\subseteq B_i$. Then $(\bigcap_{i\in I}B_i)\Gamma H^n\subseteq\bigcap_{i\in I}B_i$ and $(\bigcap_{i\in I}B_i]\subseteq\bigcap_{i\in I}(B_i]=\bigcap_{i\in I}B_i$. Therefore, $\bigcap_{i\in I}B_i$ is an ordered n-right-Γ-hyperideal of H.

Lemma 2.1.2. Let H be an ordered Γ-semihypergroup and Q a non-empty subset of H. Then the following statements hold.

(i) $(H^m\Gamma Q]$ is an ordered m-left-Γ-hyperideal of H.

(ii) $(Q\Gamma H^n]$ is an ordered n-right-Γ-hyperideal of H.

Proof. (i) By Lemma 1.8, we have that

$$\begin{aligned}(H^m\Gamma Q]\Gamma(H^m\Gamma Q] &\subseteq ((H^m\Gamma Q)\Gamma(H^m\Gamma Q)]\\ &\subseteq ((H^m\Gamma H)\Gamma(H^m\Gamma Q)]\\ &\subseteq (H\Gamma(H\Gamma H^{m-1}\Gamma Q)]\\ &= ((H\Gamma H)\Gamma(H^{m-1}\Gamma Q)]\\ &\subseteq (H\Gamma(H^{m-1}\Gamma Q)]\\ &= ((H\Gamma H^{m-1})\Gamma Q)]\\ &= (H^m\Gamma Q].\end{aligned}$$

Hence, $(H^m\Gamma Q]$ is a sub-Γ-semihypergroup of H. We see that

$$\begin{aligned}H^m\Gamma(H^m\Gamma Q] &\subseteq H\Gamma(H\Gamma H^{m-1}\Gamma Q]\\ &= (H]\Gamma(H\Gamma H^{m-1}\Gamma Q]\\ &\subseteq (H\Gamma(H\Gamma H^{m-1}\Gamma Q)]\\ &= ((H\Gamma H)\Gamma(H^{m-1}\Gamma Q)]\\ &\subseteq (H\Gamma(H^{m-1}\Gamma Q)]\\ &= ((H\Gamma H^{m-1})\Gamma Q)]\end{aligned}$$

$$=(H^m\Gamma Q],$$

and $((H^m\Gamma Q]]=(H^m\Gamma Q]$. Therefore, $(H^m\Gamma Q]$ is an ordered m-left-Γ-hyperideal of H. The case (ii) can be proved similarly (i).

Lemma 2.1.3. Let H be an ordered Γ-semihypergroup. Then the following statements hold.

(i) Every ordered m-left-Γ-hyperideal is an ordered (m,n)-quasi-Γ-hyperideal of H for all positive integer n.

(ii) Every ordered n-right-Γ-hyperideal is an ordered (m,n)-quasi-Γ-hyperideal of H for all positive integer m.

Proof. (i) Assume that A is an ordered m-left-Γ-hyperideal of H and let n be a positive integer. Then A is a sub Γ-semihypergroup of H. Thus $(H^m\Gamma A]\cap(A\Gamma H^n]\subseteq(H^m\Gamma A]\subseteq(A]=A$ and $(A]=A$. Therefore, A is an ordered (m,n)-quasi-Γ-hyperideal of H for all positive integer n. The case (ii) can be proved similarly (i).

Theorem 2.1.4. Let H be an ordered Γ-semihypergroup and A an ordered m-left-Γ-hyperideal and B an ordered n-right-Γ-hyperideal of H. Then $A\cap B=\varnothing$ or $A\cap B$ is an ordered (m,n)-quasi-Γ-hyperideal of H.

Proof. Assume that $A\cap B\neq\varnothing$. Since A and B are sub-Γ-semihypergroups of H, we have $A\cap B$ is a sub-Γ-semihypergroup of H. We see that

$$\begin{aligned}(H^m\Gamma(A\cap B)]\cap((A\cap B)\Gamma H^n] &\subseteq(H^m\Gamma A]\cap(B\Gamma H^n]\\ &\subseteq(A]\cap(B]\\ &=A\cap B.\end{aligned}$$

and $(A\cap B]\subseteq(A]\cap(B]=A\cap B$. Hence, $A\cap B$ is an ordered (m,n)-quasi-Γ-hyperideal of H.

Definition 2.1.5. A sub-Γ-semihypergroup Q of an ordered Γ-semihypergroup H has the ordered (m,n)-Γ-intersection property if Q is the intersection of an ordered m-left-Γ-hyperideal and an ordered n-right-Γ-hyperideal of H.

Theorem 2.1.6. Let H be an ordered Γ-semihypergroup and Q an ordered (m,n)-quasi-Γ-hyperideal of H. Then following statements are equivalent.

(i) Q has the ordered (m,n)-Γ-intersection property.

(ii) $(Q\cup H^m\Gamma Q]\cap(Q\cup Q\Gamma H^n]=Q$.

(iii) $(H^m\Gamma Q]\cap(Q\cup Q\Gamma H^n]\subseteq Q$.

(iv) $(Q\cup H^m\Gamma Q]\cap(Q\Gamma H^n]\subseteq Q$.

Proof. $(i)\rightarrow(ii)$ Assume that Q has the ordered (m,n)-Γ-intersection property. Since

$$Q\subseteq Q\cup(H^m\Gamma Q]=(Q]\cup(H^m\Gamma Q]=(Q\cup(H^m\Gamma Q)]$$

and

$$Q\subseteq Q\cup(Q\Gamma H^n]=(Q]\cup(Q\Gamma H^n]=(Q\cup(Q\Gamma H^n)],$$

we have $Q\subseteq(Q\cup H^m\Gamma Q]\cap(Q\cup(Q\Gamma H^n]$.

Since Q has the ordered (m,n)-Γ-property, there exist an ordered m-left-Γ-hyperideal A and ordered n-right-Γ-hyperideal B of H, such that $Q=A\cap B$. This implies that $Q\subseteq A$ and $Q\subseteq B$, so $(H^m\Gamma Q]\subseteq(H^m\Gamma A]\subseteq(A]=A$ and $(Q\Gamma H^n]\subseteq(B\Gamma H^n]\subseteq(B]=B$.

Thus $(Q\cup H^m\Gamma Q]=(Q]\cup(H^m\Gamma Q]=Q\cup(H^m\Gamma Q]\subseteq A$ and $(Q\cup Q\Gamma H^n]=(Q]\cup(Q\Gamma H^n]=Q\cup(Q\Gamma H^n]\subseteq B$.

Hence, $(Q\cup H^m\Gamma Q]\cap(Q\cup Q\Gamma H^n]\subseteq A\cap B=Q$.

Therefore, $(Q\cup H^m\Gamma Q]\cap(Q\cup Q\Gamma H^n]=Q$.

$(ii)\rightarrow(i)$ Assume that $(Q\cup H^m\Gamma Q]\cap(Q\cup Q\Gamma H^n]=Q$. We shall show that $(Q\cup H^m\Gamma Q]$ is an ordered m-left-Γ-hyperideal and $(Q\cup Q\Gamma H^n]$ an ordered n-right-Γ-hyperideal of H. By Lemma 2.1.2, we have $(H^m\Gamma Q]$ is an ordered m-left-Γ-hyperideal and $(Q\Gamma H^n]$ an ordered n-right-Γ-hyperideal of H and so $(H^m\Gamma Q]$ and $(Q\Gamma H^n]$ are sub-Γ-semihypergroup of H. We see that

$$\begin{aligned}&(Q\cup H^m\Gamma Q]\Gamma(Q\cup H^m\Gamma Q]\\ &=(Q\cup(H^m\Gamma Q])\Gamma(Q\cup(H^m\Gamma Q])\\ &=(Q\Gamma Q)\cup(H^m\Gamma Q]\Gamma Q\cup Q\Gamma(H^m\Gamma Q]\cup(H^m\Gamma Q]\Gamma(H^m\Gamma Q]\\ &\subseteq(Q\Gamma Q)\cup(H^m\Gamma Q]\Gamma(Q]\\ &\quad\cup(H]\Gamma(H^m\Gamma Q]\cup(H^m\Gamma Q]\Gamma(H^m\Gamma Q]\\ &\subseteq(Q\Gamma Q)\cup(H^m\Gamma Q\Gamma Q]\cup(H\Gamma H^m\Gamma Q]\cup(H^m\Gamma Q\Gamma H^m\Gamma Q]\\ &\subseteq Q\cup(H^m\Gamma Q]\cup(H^m\Gamma Q]\cup(H^m\Gamma Q]\\ &=Q\cup(H^m\Gamma Q]\\ &=(Q\cup H^m\Gamma Q].\end{aligned}$$

Thus $(Q\cup H^m\Gamma Q]$ is a sub-Γ-semihypergroup of H. So, we have

$$\begin{aligned}H^m\Gamma(Q\cup H^m\Gamma Q] &=H^m\Gamma(Q\cup(H^m\Gamma Q])\\ &=H^m\Gamma Q\cup H^m\Gamma(H^m\Gamma Q]\\ &\subseteq H^m\Gamma Q\cup(H^m\Gamma Q] \text{ (By Lemma 2.1.2)}\\ &=(H^m\Gamma Q]\\ &\subseteq(Q]\cup(H^m\Gamma Q]\end{aligned}$$

$= (Q \cup H^m \Gamma Q]$,
and $((Q \cup H^m \Gamma Q]] = (Q \cup H^m \Gamma Q]$. Hence $(Q \cup H^m \Gamma Q]$ is an ordered m-left-Γ-hyperideal of H.

In a same way, we can proof that $(Q \cup Q\Gamma H^n]$ is an ordered n-right-Γ-hyperideal of H. We see that $(Q \cup Q\Gamma H^n]\Gamma(Q \cup Q\Gamma H^n]$
$= (Q \cup (Q\Gamma H^n])\Gamma(Q \cup (Q\Gamma H^n])$
$= (Q\Gamma Q) \cup Q\Gamma(Q\Gamma H^n] \cup (Q\Gamma H^n]\Gamma Q \cup (Q\Gamma H^n]\Gamma(Q\Gamma H^n]$
$\subseteq (Q\Gamma Q) \cup (Q]\Gamma(Q\Gamma H^n] \cup (Q\Gamma H^n]\Gamma(H] \cup (Q\Gamma H^n]\Gamma(Q\Gamma H^n]$
$\subseteq Q\Gamma Q \cup (Q\Gamma Q\Gamma H^n] \cup (Q\Gamma H^n \Gamma H] \cup (Q\Gamma H^n \Gamma Q\Gamma H^n]$
$\subseteq Q \cup (Q\Gamma H^n] \cup (Q\Gamma H^n] \cup (Q\Gamma H^n]$
$= Q \cup (Q\Gamma H^n]$
$= (Q \cup Q\Gamma H^n]$.
Thus $(Q \cup Q\Gamma H^n]$ is a sub-Γ-semihypergroup of H.
So, we have
$(Q \cup Q\Gamma H^n]\Gamma H^n = (Q \cup (Q\Gamma H^n])\Gamma H^n$
$= Q\Gamma H^n \cup (Q\Gamma H^n]\Gamma H^n$
$\subseteq Q\Gamma H^n \cup (Q\Gamma H^n]$ (By Lemma 2.1.2)
$= (Q\Gamma H^n]$
$\subseteq (Q] \cup (Q\Gamma H^n]$
$= (Q \cup Q\Gamma H^n]$,
and $((Q \cup Q\Gamma H^n]] = (Q \cup Q\Gamma H^n]$. Hence, $(Q \cup Q\Gamma H^n]$ is an ordered n-right-Γ-hyperideal of H. Therefore, Q has the ordered (m,n)-Γ-intersection property.

$(ii) \rightarrow (iii)$ Assume that $(Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] = Q$. Since $(H^m \Gamma Q] \subseteq (Q] \cup (H^m \Gamma Q] = (Q \cup H^m \Gamma Q]$, it follows that $(H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] \subseteq (Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] = Q$. Hence, $(H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] \subseteq Q$.

$(iii) \rightarrow (ii)$ Assume that $(H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] \subseteq Q$. Since $Q \subseteq Q \cup (H^m \Gamma Q] = (Q \cup H^m \Gamma Q]$ and $Q \subseteq Q \cup (Q\Gamma H^n] = (Q \cup Q\Gamma H^n]$, it follows that $Q \subseteq (Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n]$. We see that $(Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n]$
$= (Q \cup (H^m \Gamma Q]) \cap (Q \cup (Q\Gamma H^n])$
$= (Q \cap (Q \cup (Q\Gamma H^n])) \cup ((H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n])$
$\subseteq Q \cup Q$
$= Q$.
Therefore, $(Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] = Q$.

$(ii) \rightarrow (iv)$ Assume that $(Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] = Q$. Since $(Q\Gamma H^n] \subseteq (Q] \cup (Q\Gamma H^n] = (Q \cup Q\Gamma H^n]$, it follows that $(Q \cup H^m \Gamma Q] \cap (Q\Gamma H^n] \subseteq (Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] = Q$. Hence, $(Q \cup H^m \Gamma Q] \cap (Q\Gamma H^n] \subseteq Q$.

$(iv) \rightarrow (ii)$ Assume that $(Q \cup H^m \Gamma Q] \cap (Q\Gamma H^n] \subseteq Q$. Since $Q \subseteq Q \cup (Q\Gamma H^n] = (Q \cup Q\Gamma H^n]$ and $Q \subseteq Q \cup (H^m \Gamma Q] = (Q \cup H^m \Gamma Q]$, it follows that $Q \subseteq (Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n]$.
We see that $(Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n]$
$= (Q \cup (H^m \Gamma Q]) \cap (Q \cup (Q\Gamma H^n])$
$= ((Q \cup (Q\Gamma H^n]) \cap Q) \cup ((Q \cup (H^m \Gamma Q]) \cap (Q\Gamma H^n])$
$\subseteq Q \cup Q$
$= Q$.
Therefore, $(Q \cup H^m \Gamma Q] \cap (Q \cup Q\Gamma H^n] = Q$.

Lemma 2.1.7. Every ordered m-left-Γ-hyperideal and ordered n-right-Γ-hyperideal of an ordered Γ-semihypergroup have the ordered (m,n)-Γ-intersection property.

Proof. Let A be an ordered m-left-Γ-hyperideal and B an ordered n-right-Γ-hyperideal of an ordered Γ-semihypergroup H. Since A is an ordered m-left-Γ-hyperideal of H, by lemma 2.1.3 (i),
(i) we have that A is an ordered (m,n)-quasi-Γ-hyper-ideal of H. So, we have
$(H^m \Gamma A] \cap (A \cup A\Gamma H^n] = (H^m \Gamma A] \cap (A \cup (A\Gamma H^n])$
$= ((H^m \Gamma A] \cap A) \cup ((H^m \Gamma A] \cap (A\Gamma H^n])$
$\subseteq A \cup A$
$= A$.
By Theorem 2.1.6, we have that A has the ordered (m,n)-Γ-intersection property. Next, we will show that B has the ordered (m,n)-Γ-intersection property. Since B is an ordered n-right-Γ-hyper-ideal of H, by Theorem 2.1.3,
(ii) we have that B is an ordered (m,n)-quasi-Γ-hyperideal of H. So, we have $(B \cup H^m \Gamma B] \cap (B\Gamma H^n]$
$= (B \cup (H^m \Gamma B]) \cap (B\Gamma H^n]$
$= (B \cap (B\Gamma H^n]) \cup ((H^m \Gamma B] \cap (B\Gamma H^n])$
$\subseteq B \cup B$
$= B$.
By Theorem 2.1.6, we have that B has the ordered (m,n)-Γ-intersection property.

Proposition 2.1.8. Let H be an ordered Γ-semihypergroup and Q an ordered (m,n)-quasi-Γ-hyperideal of H. If $H^m \Gamma Q \subseteq Q\Gamma H^n$ or $Q\Gamma H^n \subseteq H^m \Gamma Q$, then Q has the ordered (m,n)-Γ-intersection property.

Proof. Assume that $H^m\Gamma Q \subseteq Q\Gamma H^n$. It is evident that $(H^m\Gamma Q] \subseteq (Q\Gamma H^n]$. Since Q is an ordered (m,n)-quasi-Γ-hyperideal of H, we get $H^m\Gamma Q \subseteq (H^m\Gamma Q] = (H^m\Gamma Q] \cap (Q\Gamma H^n] \subseteq Q$. This means that Q is an ordered m-left-Γ-hyperideal of H. By Lemma 2.1.7, we have that Q has the ordered (m,n)-Γ-intersection property. Similarly, we obtain $Q\Gamma H^n \subseteq H^m\Gamma Q$. Then $(Q\Gamma H^n] \subseteq (H^m\Gamma Q]$. Since Q is an ordered (m,n)-quasi-Γ-hyperideal of H, we get $Q\Gamma H^n \subseteq (Q\Gamma H^n] = (H^m\Gamma Q] \cap (Q\Gamma H^n] \subseteq Q$. This means that Q is an ordered n-right-Γ-hyperideal of H. By Lemma 2.1.7, we have that Q has the ordered (m,n)-Γ-intersection property. This completes the proof.

2.2. Ordered (m,n)-quasi-Γ-hyperideals in Regular Ordered Γ-semihypergroups

We have investigated in the previous section that every ordered m-left-Γ-hyperideal and ordered n-right-Γ-hyperideal of an ordered Γ-semihypergroup have the ordered (m,n)-Γ-intersection property, but not for ordered (m,n)-quasi-Γ-hyperideals in ordered Γ-semihypergroups. In this section, we will prove that every ordered (m,n)-quasi-Γ-hyperideal of a regular ordered Γ-semihypergroup has the ordered (m,n)-Γ-intersection property.

Definition 2.2.1. (Omidi. & Davvaz, 2017) An element a of an ordered Γ-semihypergroup H is regular if there exist $x \in H$ and $\alpha,\beta \in \Gamma$, such that $a \le a\alpha x\beta a$. This is equivalent to saying that $a \in (a\Gamma H\Gamma a]$, for every $a \in H$ or $A \in (A\Gamma H\Gamma A]$, for every $A \subseteq H$.

Lemma 2.2.2. Let H be a regular ordered Γ-semihypergroup and A a non-empty subset of H. Then the following statements hold.

(i) $A \subseteq (H^m\Gamma A]$ for all positive integer m.

(ii) $A \subseteq (A\Gamma H^n]$ for all positive integer n.

Proof. (i) Let $a \in A$. Since H is regular, there exists $x \in H$ and $\alpha,\beta \in \Gamma$ such that $a \le a\alpha x\beta a$. Since $a\alpha x \subseteq H$, it follows that $a \le a\alpha x\beta a = (a\alpha x)\beta a \subseteq H\Gamma A$ and so $A \subseteq (H\Gamma A]$. Let m be a positive integer such that $A \subseteq (H^m\Gamma A]$. Then we have $H\Gamma A \subseteq H\Gamma(H^m\Gamma A] = (H]\Gamma(H^m\Gamma A] \subseteq (H\Gamma(H^m\Gamma A)] = (H^{m+1}\Gamma A]$. Hence $A \subseteq (H^{m+1}\Gamma A]$. Therefore, $A \subseteq (H^m\Gamma A]$ for all positive integer m.

(ii) Let $a \in A$. Since H is regular, there exists $x \in H$ and $\alpha,\beta \in \Gamma$ such that $a \le a\alpha x\beta a$. Since $x\beta a \subseteq H$, it follows that $a \le a\alpha x\beta a = a\alpha(x\beta a) \subseteq A\Gamma H$ and so $A \subseteq (A\Gamma H]$. Let n be a positive integer such that $A \subseteq (A\Gamma H^n]$. Then we have $A\Gamma H \subseteq (A\Gamma H^n]\Gamma H = (A\Gamma H^n]\Gamma(H] \subseteq ((A\Gamma H^n)\Gamma H] = (A\Gamma H^{n+1}]$. Hence $A \subseteq (A\Gamma H^{n+1}]$. Therefore, $A \subseteq (A\Gamma H^n]$ for all positive integer n.

Theorem 2.2.3. Every ordered (m,n)-quasi-Γ-hyperideal of a regular ordered Γ-semihypergroup has the ordered (m,n)-Γ-intersection property.

Proof. Let Q be an ordered (m,n)-quasi-Γ-hyperideal of a regular ordered Γ-semihypergroup H. By Lemma 2.2.2, we have $Q \subseteq (Q\Gamma H^n]$ and so $(Q \cup Q\Gamma H^n] = Q \cup (Q\Gamma H^n] = (Q\Gamma H^n]$. Thus $(H^m\Gamma Q] \cap (Q \cup Q\Gamma H^n] = (H^m\Gamma Q] \cap (Q\Gamma H^n] \subseteq Q$. By Theorem 2.1.6, we have that Q has the ordered (m,n)-Γ-intersection property.

Theorem 2.2.4. Let H be a regular ordered Γ-semihypergroup and A a non-empty subset of H. Then A is an ordered (m,n)-quasi-Γ-hyperideal of H if and only if $A = (H^m\Gamma A] \cap (A\Gamma H^n]$.

Proof. Assume that A is an ordered (m,n)-quasi-Γ-hyperideal of H. Then $(H^m\Gamma A] \cap (A\Gamma H^n] \subseteq A$. By Lemma 2.2.2 we have $A \subseteq (H^m\Gamma A]$ and $A \subseteq (A\Gamma H^n]$ and so $A \subseteq (H^m\Gamma A] \cap (A\Gamma H^n]$. Therefore, $A = (H^m\Gamma A] \cap (A\Gamma H^n]$.

Conversely, assume that $A = (H^m\Gamma A] \cap (A\Gamma H^n]$. By Theorem 2.1.2, we have $(H^m\Gamma A]$ is an ordered m-left-Γ-hyperideal and $(A\Gamma H^n]$ is an ordered n-right-Γ-hyperideal of H. By Theorem 2.1.4, we have that A is an ordered (m,n)-quasi-Γ-hyperideal of H.

3. Conclusions

The results of the resecrch would be that every ordered m-left-Γ-hyperideal and ordered n-right-Γ-hyperideal of ordered Γ-semihypergroups have the ordered (m,n)-Γ-intersection property, but not for ordered (m,n)-quasi-Γ-hyperideals in ordered-Γ-semihypergroups. We have added some properties of the ordered-Γ-semihypergroups, resulting in every ordered (m,n)-quasi-Γ-hyperideals of a regular ordered-Γ-semihypergroup having the ordered (m,n)-Γ-intersection property.

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