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The Analytical Solutions of Bateman-Burgers Equation

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Abstract

The objective of this study is to find the new method to get the analytical solutions of nonlinear partial differential equation, namely Bateman-Burgers equation which have the form $u_t + uu_x = \upsilon u_{xx}$. The simple equation method is chosen to find the answer. The results of the study show that this method is effective at achieving the solutions of the Bateman-Burgers equation with both Bernoulli equation and Riccati equation.

Keywords: Simple equation method, Nonlinear partial differential equation, Bernoulli equation, Riccati equation

Introduction

Nonlinear partial differential equations are very important in both mathematics and physics. They describe a lot of systems in terms of mathematics equation. In 1915, Harry Bateman proposed the equation which is well-known in many areas of applied mathematics such as gas dynamics, fluid mechanics and traffic flow. The equation was studied later in 1948 by Jan Burgers[1]. So, we got the Bateman-Burgers equation in the following form

$$u_t + uu_x = \upsilon u_{xx}, \qquad (1)$$

where u is a function of variables x and t, υ is the viscosity of a fluid[2].

The research objective is to use the simple equation method (SE method) with both the Bernoulli equation and the Riccati equation to solve the analytical solutions of the Bateman-Burgers equation. Next, we will introduce the simple equation method.

Simple Equation Method

In this section, the process of simple equation method will be shown[3-5].

Step 1: Given the nonlinear partial differential equation in the form

$$P(u,u_t,u_x,u_{xx},u_{xt},...)=0, \quad (2)$$

where u is the function of x and t.

Step 2: To transform $u(x,t)=U(\xi)$ we set $\xi=x-bt$ and use this to transform equation (2) into an ordinary differential equation

$$G(U,U_\xi,U_{\xi\xi},U_{\xi\xi\xi},...)=0, \quad (3)$$

where b is nonzero constant of wave velocity and G is a polynomial of $U(\xi)$ and its derivatives.

Step 3: The solution of equation (3) can be express in the form

$$U(\xi)=\sum_{i=0}^{N}a_iF^i, \quad (4)$$

where $F=F(\xi)$, a_i are constant and $a_N\neq 0$.

Step 4: Find the value of N by balancing between the highest order derivative and the nonlinear terms.

Step 5: For Bernoulli equation we use

$$F_\xi=cF(\xi)+dF^2(\xi), \quad (5)$$

where c and d are nonzero constant.

For Riccati equation we use

$$F_\xi=\alpha F^2(\xi)+\beta, \quad (6)$$

where α and β are nonzero constant.

Step 6: Substituting N from step 4 into equation (4) and then collect all terms which have the same power of F and set them to zero, the solution of equation (5) and equation (6) are described in two cases.

For equation (5) Bernoulli equation

Case 1: $c>0$, $d<0$, ξ_0 and υ are constant,

$$F(\xi)=\frac{ce^{[c(\xi+\xi_0)]}}{1-de^{[c(\xi+\xi_0)]}}. \quad (7)$$

Case 2: $c<0$, $d>0$, ξ_0 and υ are constant,

$$F(\xi)=-\frac{ce^{[c(\xi+\xi_0)]}}{1+de^{[c(\xi+\xi_0)]}}. \quad (8)$$

For equation (6) Riccati equation

Case 1: $\alpha\beta<0$, $\xi_0>0$ and $v=\pm 1$,

$$F(\xi)=\frac{-\sqrt{-\alpha\beta}}{\alpha}\tanh\left(\sqrt{-\alpha\beta}\,\xi-\frac{v\ln(\xi_0)}{2}\right). \quad (9)$$

Case 2: $\alpha\beta>0$ and ξ_0 is a constant,

$$F(\xi)=\frac{\sqrt{\alpha\beta}}{\alpha}\tan\left(\sqrt{\alpha\beta}(\xi+\xi_0)\right). \quad (10)$$

Solutions of Bateman-Burgers equation with Bernoulli equation case

To transform equation (1) into ordinary differential equation (ODE) we set the wave variable $\xi=x-bt$ where b is nonzero constant of wave [6-8] velocity so we get

$$-bU_\xi+UU_\xi-\upsilon U_{\xi\xi}=0, \quad (11)$$

the solution of equation (11) is defined by equation (4).

Find N by balancing between the highest order derivative and the nonlinear terms

$$N+2=N+N+1$$

$$N=1$$

there for equation (4) will be

$$U(\xi)=a_0+a_1F. \quad (12)$$

Differentiating

$$U_\xi=a_1cF+a_1dF^2, \quad (13)$$

$$U_{\xi\xi}=a_1c^2F+3a_1cdF^2+2a_1d^2F^3, \quad (14)$$

$$UU_\xi=a_0a_1cF+a_0a_1dF^2+a_1^2cF^2+a_1^2dF^3. \quad (15)$$

Substituting equations (13), (14) and (15) into equation (11), we obtain

$$\begin{aligned}&-ba_1cF-ba_1dF^2+a_0a_1cF+a_0a_1dF^2\\&+a_1^2cF^2+a_1^2dF^3-\upsilon a_1c^2F\\&-3\upsilon a_1cdF^2-2\upsilon a_1d^2F^3=0.\end{aligned} \quad (16)$$

Collect all terms which have the same power of F and set to zero

$$F^1:\ -ba_1c+a_0a_1c-\upsilon a_1c^2=0, \quad (17)$$

$$F^2:\ -ba_1d+a_0a_1d+a_1^2c-3\upsilon a_1cd=0, \quad (18)$$

$$F^3:\ a_1^2d-2\upsilon a_1d^2=0. \quad (19)$$

Solving the system of equations (17), (18) and (19), we get

$$a_1=2\upsilon d,\ b=a_0-\upsilon c. \quad (20)$$

Substituting equations (7), (8) and (20) into equation (12), the solutions of the Bateman-Burgers equation may be considered as,

Case 1: $c>0, d<0$,

$$u(x,t)=a_0+\frac{2\upsilon dce^{[c(x-(a_0-\nu c)t+\xi_0)]}}{1-de^{[c(x-(a_0-\nu c)t+\xi_0)]}}. \quad (21)$$

Case 2: $c<0, d>0$,

$$u(x,t)=a_0-\frac{2\upsilon dce^{[c(x-(a_0-\nu c)t+\xi_0)]}}{1+de^{[c(x-(a_0-\nu c)t+\xi_0)]}}. \quad (22)$$

For case 1, using parameters $c=1, d=-1, 1\le x\le 15, 1\le t\le 15, \xi_0=0, a_0=0$ and $\upsilon=1$ the solutions of the Bateman-Burgers equation can be demonstrated as in figure 1.

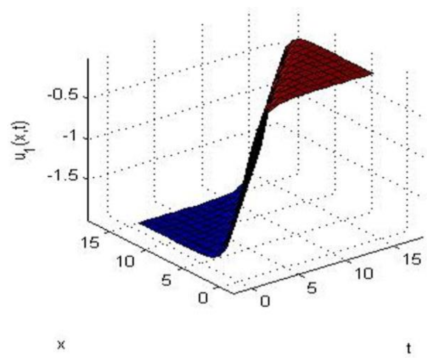

Figure 1. The solutions by the SE method with Bernoulli equation in case 1.

In the second case, using parameters $c=-10, d=10, 1\le x\le 15, 1\le t\le 15, \xi_0=0, a_0=0$ and $\upsilon=1$ the solutions of the Bateman-

Burgers equation can be demonstrated as in figure 2.

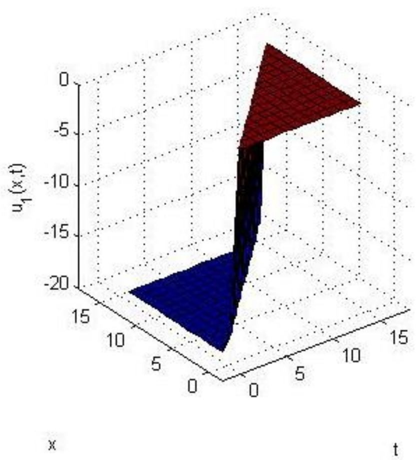

Figure 2. The solutions by the SE method with Bernoulli equation in case 2.

Solutions of Bateman-Burgers equation with Riccati equation case

For Riccati equation, we use F_ξ from equation (6), since we consider the same equation then $N=1$ and $U(\xi)$ is the same as equation (12).

Differentiating

$$U_\xi = a_1\alpha F^2 + a_1\beta\,, \tag{23}$$

$$U_{\xi\xi} = 2a_1\alpha^2 F^3 + 2a_1\alpha\beta F\,, \tag{24}$$

$$UU_\xi = a_0a_1\alpha F^2 + a_0a_1\beta + a_1^2\alpha F^3 + a_1^2\beta F\,. \tag{25}$$

Substituting equations (23), (24) and (25) into equation (11), we obtain

$$\begin{aligned}&-ba_1\alpha F^2 - ba_1\beta + a_0a_1\alpha F^2\\ &+a_0a_1\beta + a_1^2\alpha F^3 + a_1^2\beta F\\ &-2\upsilon a_1\alpha^2 F^3 - 2\upsilon a_1\alpha\beta F = 0.\end{aligned} \tag{26}$$

Collect all terms which have the same power of F and set to zero

$$F^0:\ -ba_1\beta + a_0a_1\beta = 0\,, \tag{27}$$

$$F^1:\ a_1^2\beta - 2\upsilon a_1\alpha\beta = 0\,, \tag{28}$$

$$F^2:\ -ba_1\alpha + a_0a_1\alpha = 0\,, \tag{29}$$

$$F^3:\ a_1^2\alpha - 2\upsilon a_1\alpha^2 = 0\,. \tag{30}$$

Solving the system of equations (27), (28), (29) and (30), we get

$$a_1 = 2\upsilon\alpha\,,\ \ b = a_0\,. \tag{31}$$

Substituting equations (9), (10) and (31) into equation (12), the solutions of the Bateman-Burgers equation may be considered as,

Case 1 : $\alpha\beta < 0$, $\xi_0 > 0$ and $v = \pm 1$,

$$u(x,t) = b - 2\upsilon\sqrt{-\alpha\beta}\tanh\left(\sqrt{-\alpha\beta}(x-bt) - \frac{v\ln(\xi_0)}{2}\right). \tag{32}$$

Case 2 : $\alpha\beta > 0$ and ξ_0 is a constant,

$$u(x,t) = b + 2\upsilon\sqrt{\alpha\beta}\tan\left(\sqrt{\alpha\beta}(x-bt+\xi_0)\right). \tag{33}$$

For case 1, using parameters $\alpha=1, \beta=-1, 1\le x\le 15, 1\le t\le 15, \xi_0=1, b=1$ and $\upsilon=1$ the solutions of the Bateman-Burgers equation can be demonstrated as in figure 3.

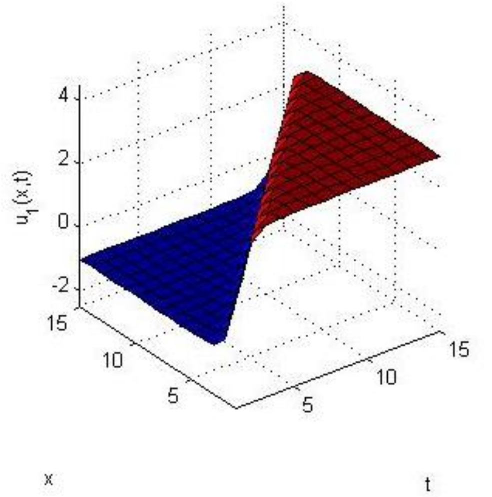

Figure 3. The solutions by the SE method with Riccati equation in case 1.

In the second case, using parameters $\alpha=1, \beta=1, 1\le x\le 15, 1\le t\le 15, \xi_0=1, b=1$ and $\upsilon=1$ the solutions of the Bateman-Burgers equation can be demonstrated as in figure 4.

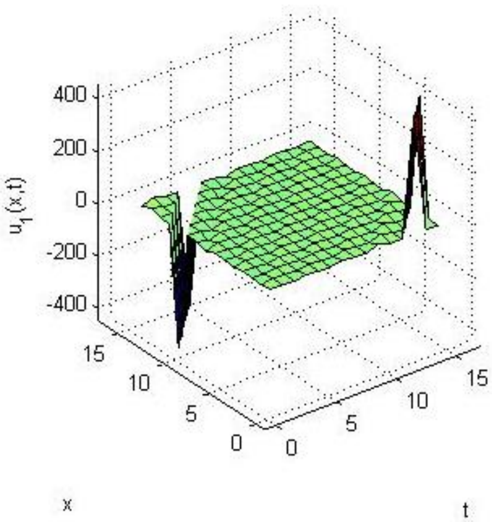

Figure 4. The solutions by the SE method with Riccati equation in case 2.

Conclusions

The simple equation method is applied to solve the Bateman-Burgers equation with the wave variable $\xi = x - bt$ in both Bernoulli equation and Riccati equation. The solutions may be defined in equation (4) along with the help from equation (5), (7) and (8) for Bernoulli equation and equation (6), (9) and (10) for Riccati equation. After some balancing and calculating, the solutions are achieved in equation (21) and (22) for Bernoulli equation and equation (32) and (33) for Riccati equation. The simple equation in both Bernoulli equation and Riccati equation shows that this method in both cases is effective for solving Bateman-Burgers equation.

References

1. Burgers, JM. A mathematical model illustrating the theory of turbulence. Adv Appl Mech 1948;1:171-99.
2. Bateman H. Some recent researches on the motion of fluids. Mon Weather Rev 1915;43:163-70.
3. Boateng K, Yang W, Otoo M, Yaro D. Dispersive traveling wave solution for non-linear waves dynamical models. J appl math phys 2019;7:2467-80.
4. Vitanov N, Dimitrova Z, Vitanov K. Simple Equations Method (SEsM): Algorithm, connection with Hirota method, inverse

scattering transform method, and several other methods. Entropy 2021;23:1-36.
5. Nofal A. Simple equation method for nonlinear partial differential equations and its applications. J Egyptian Math Soc 2015;24:204-9.
6. Ablowitz J, Clarkson A. Soliton, nonlinear evolution equations and inverse scatting. London Mathematical Society Lecture Note Series 149. 1st ed. Cambridge University Press; 1991.
7. Kudryashov A. Exact soliton solutions of the generalized evolution equation of wave dynamics. J Appl Math Mech 1988;52:361-5.
8. Dechanubeksa C, Chinviriyasit S. New analytical solutions of (1+1) dimension Chaffee-Infante equation using modified simple equation method. Proceedings of the 14th IMT-GT ICMSA; 2018 Dec 8-10; Songkhla,Thailand. Thaksin University; 2018.