Improving of Test Statistic for the Risk Ratio in a Correlated 2 x 2 Table with Structural Zero

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Abstract

The purpose of this research was to develop test statistic for the risk ratio in a correlated 2 x 2 table with structural zero when sample size is small. We studied Wald test statistic, Logarithmic transformation test statistic, Fieller's test statistic, Rao's score test statistic and improving Wald test statistic. Also, we consider the performance of hypothesis testing for the risk ratio by power of the test and type I error rate close to the significant level. Simulation studies suggest improving Wald test statistic has type I error rate closest to significant level and powerful test when risk ratio value in alternative hypotheses are less.

Keywords: Power of tests, Test of hypothesis, Wald test

Introduction

The problem of correlated 2 x 2 table with a structural zero in one of the off diagonal cells, the structural zero means that it is theoretically impossible to observe for a particular cell sometimes appear in infection disease studies and two-step procedure studies. A typical example of calves; calves were first classified according to whether they got a primary pneumonia infection and then reclassified according to whether they developed a secondary infection within a certain time period after the first infection cleared up. In this case, the interest in evaluating the risk ratio between a secondary infection (p_{11}), given a primary infection and the primary infection ($p_{1.}$)[1], the responses taken from the same group of calves are not independent in Table 1.

Table 1 Example and probability of Agresti

Primary infection	Secondary infection		Total
	Yes	No	
Yes	n11 = 30 (p11)	n12 = 63 (p12)	$n_{1.}$ = 93 ($p_{1.}$)
No	n21 = 0	n22 = 63 (p22)	$n_{2.}$ = 63 ($p_{2.}$)
Total	n11 = 30 (p11)	n2. = 126 (p.2)	n = 156 (1)

Lui discussed the estimation of the risk ratio; he developed three asymptotic interval estimators using Wald test statistic, the logarithmic transformation and Fieller's theorem. He concluded that when the probability of primary infection is small or moderate, the interval estimator using the logarithmic transformation outperform the two estimators when the sample size does not exceed 100[2]. Gupta and Tian further studied confidence intervals for the risk ratio from Lui and derived a fourth confidence interval base on Rao's score test. In addition, they compared performance of four test statistics and concluded that the confidence interval estimator using the Rao's score test and the logarithmic transformation outperforming the other two estimators. In addition, the Rao's score test statistic is more consistent than the other test statistics[3].

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This research is interested in study performance of test statistic for the risk ratio in a correlated 2 x 2 table with structural zero. We propose an improving Wald test statistic additional to Gupta and Tian when sample size is small.

Test statistic for the risk ratio

Consider a random sample of n subjects in table 1 is assumed to be trinomial distribution.

$$f(n_{11},n_{12},n_{22})=\frac{n!}{n_{11}!n_{12}!n_{22}!}p_{11}^{n_{11}}p_{12}^{n_{12}}p_{22}^{n_{22}} \quad (1)$$

Then, the estimators of the probabilities are $\hat{p}_{11}=n_{11}/n$, $\hat{p}_{12}=n_{12}/n$ and $\hat{p}_{22}=n_{22}/n$

Also $p_{1.}=p_{11}+p_{12}, p_{11}+p_{12}+p_{22}$ The risk ratio between a secondary infection, given a primary infection and the primary infection is defined as $\mathrm{RR}=(p_{11}/p_{1.})/p_{1.}=p_{11}/p_{1.}^2=\varphi$. And hypothesis testing for the risk ratio is $\mathrm{H}_0:\varphi=1$ versus $\mathrm{H}_1:\varphi\neq 1$.

Wald test statistic:

$$\mathrm{T_w}=\frac{\sqrt{n}(\hat{\varphi}-\varphi)}{\sqrt{\hat{\mathrm{Var}}(\hat{\varphi})}}=\frac{\sqrt{n}\left(\frac{\hat{p}_{11}}{\hat{p}_{1.}^2}-\varphi\right)}{\sqrt{\frac{\hat{p}_{11}(1-\hat{p}_{11})}{\hat{p}_{1.}^4}}} \quad (2)$$

We will reject null hypothesis when $|\mathrm{T_W}|\geq \mathrm{Z}_{1-\alpha/2}$ where $\mathrm{Z}_{1-\alpha/2}$ is the 100 $100(1-\alpha/2)$ percentile point of the standard normal distribution. Logarithmic transformation test statistic:

$$\mathrm{T_L}=\frac{\sqrt{n}(\ln(\hat{\varphi})-\ln(\varphi))}{\sqrt{\hat{\mathrm{Var}}(\hat{\varphi})}}=\frac{\sqrt{n}\left(\ln\left(\frac{\hat{p}_{11}}{\hat{p}_{1.}^2}\right)-\ln(\varphi)\right)}{\sqrt{\frac{1-\hat{p}_{11}}{\hat{p}_{11}}}} \quad (3)$$

We will reject null hypothesis when $|\mathrm{T_L}|\geq \mathrm{Z}_{1-\alpha/2}$ where $\mathrm{Z}_{1-\alpha/2}$ is the $100(1-\alpha/2)$ percentile point of the standard normal distribution.

Fieller's test statistic:

$$\mathrm{T_F}=\sqrt{n}\left(\hat{p}_{11}-\frac{\varphi\left(n\hat{p}_{1.}^2-\hat{p}_{1.}\right)}{n-1}\right)\Big/\sqrt{\hat{\mathrm{Var}}(\hat{\varphi})}$$

$$=\frac{\sqrt{n}\left(\hat{p}_{11}-\frac{\varphi\left(n\hat{p}_{1.}^2-\hat{p}_{1.}\right)}{n-1}\right)}{\sqrt{\hat{p}_{11}(1-\hat{p}_{11})+\frac{\varphi_0^2\left(2n\hat{p}_{1.}-1\right)^2\hat{p}_{1.}(1-\hat{p}_{1.})}{(n-1)^2}+\frac{2\varphi_0\left(2n\hat{p}_{1.}-1\right)\hat{p}_{11}\hat{p}_{22}}{n-1}}} \quad (4)$$

We will reject null hypothesis when $|\mathrm{T_F}|\geq \mathrm{Z}_{1-\alpha/2}$ where $\mathrm{Z}_{1-\alpha/2}$ is the $100(1-\alpha/2)$ percentile point of the standard normal distribution.

Score test statistic:

$$T_S=U^T(\varphi_0,p_{11}^0)I(\varphi_0,p_{11}^0)^{-1}U(\varphi_0,p_{11}^0)$$

$$=\left[\frac{-n_{12}}{2(1-\sqrt{\varphi_0 p_{11}^0})}+\frac{(n-n_{11}-n_{12})\sqrt{p_{11}^0}}{2(\sqrt{\varphi_0}-\sqrt{p_{11}^0})}\right]^2\frac{(1-p_{11}^0)}{np_{11}^0} \quad (5)$$

We will reject null hypothesis when $T_S>\chi^2_{1,\alpha}$ where $\chi^2_{1,\alpha}$ is upper α-percentile of the central χ^2-distribution with one degree of freedom and p_{11}^0 is solution of $U_2(\varphi,p_{11})=0$.

Improving the Wald test statistic:

In case of small sample, we have adding constant, $\mathrm{c}=0.46\mathrm{z}_{\frac{\alpha}{2}}^2/\mathrm{n}$, the tail probability of normal by following Guan[4]. The properties of the mean and the variance is $\mathrm{E}(\hat{\varphi}+\mathrm{c})=\varphi+\mathrm{c}$ and $\mathrm{Var}(\hat{\varphi}+\mathrm{c})=\mathrm{Var}(\varphi)$

$$\mathrm{T_{IW}}=\frac{\sqrt{n}\left[(\hat{\varphi}+0.46z_{\frac{\alpha}{2}}^2/n)-\varphi\right]}{\sqrt{\widehat{Var}(\hat{\varphi})}}=\frac{\sqrt{n}\left[\left(\frac{\hat{p}_{11}}{\hat{p}_{1.}^2}+0.46z_{\frac{\alpha}{2}}^2/n\right)-\varphi\right]}{\sqrt{\frac{\hat{p}_{11}(1-\hat{p}_{11})}{\hat{p}_{1.}^4}}} \quad (6)$$

We will reject null hypothesis when $|\mathrm{T_{IW}}|\geq \mathrm{Z}_{1-\alpha/2}$ where $\mathrm{Z}_{1-\alpha/2}$ is the $100(1-\alpha/2)$ percentile point of the standard normal distribution.

Methods of evaluating tests

The worth test statistic must be control the type I error with the most power of the test.

Methods

A. Generate data set according to trinomial distribution.

B. Select two sample sizes n = 25(small size) and 50 (moderate size) [2]. C. Select five parameters φ = 0.25, 0.75, 1.0, 1.25 and 1.75.

D. Select three parameters $p_{1.}$ = 0.25, 0.5 and 0.75.

F. Select nominal level (α) = 0.05

G. Compare the performance of test statistic by considering the value of empirical type I error and empirical power of the test with R program. We generate 10,000 data sets for each combination of sample size and parameter.

Results of simulation

To compare the empirical type I error value of five tests about the risk ratio test, When sample size n = 25 improving Wald test has empirical type I error value closest to significant level. Also, when sample size n = 50 Rao's score test and improving Wald test have empirical type I error value close to significant level (Table 2). In addition, when risk ratio value in alternative hypotheses are less than 1 and primary infection rate are low (0.25,0.5), the empirical power of Wald test is the most powerful and improving Wald test is the second most powerful test. When risk ratio value great than 1 and primary infection rate are low (0.25, 0.5), Fieller's test is the most powerful. When primary infection rate is high (0.75), all five tests almost the same power irrespective of the risk ratio and sample size (Table 3).

Conclusion and comments

This research is preliminary for adding the tail probability of normal for test statistic for the risk ratio in a correlated 2 x 2 table with structural zero when sample size is small. In addition to the four test statistics, the Wald test, the logarithmic transformation test, the Fieller's test and the Rao's score test, by Gupta and Tian, we have proposed an improving Wald test statistic. The performances of these five tests, in terms of empirical type I error shows the improving Wald test statistic and Rao's score test are close to significant level. When risk ratio values are less than 1 and primary infection rate are low, the power of the Wald test is the most powerful. However, when the risk ratio values in alternative hypotheses are greater than 1, the power of Fieller's test is the most powerful. We consider the improved Wald test is the most efficient when small sample size and primary infection rate are low because the empirical type I error rate of the improving Wald test is closer than the Wald test, although the power of improving Wald test is less than the Wald test but these are greater than the Rao's score test.

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Table 2 The empirical type 1 error of the tests at significant level (α) = 0.05.

n	Type 1 error					
	$p_{1.}$	T_W	T_L	T_F	T_S	T_{IW}
25	**0.25**	0.1179	0.0287	0.0239	0.0337	0.0619
	0.50	0.0811	0.0438	0.0998	0.0800	0.0614
	0.75	0.1116	0.0530	0.1821	0.0725	0.0399
50	**0.25**	0.0983	0.0359	0.0392	0.0409	0.0811
	0.50	0.0693	0.0464	0.0622	0.0605	0.0642
	0.75	0.0733	0.0510	0.1041	0.0679	0.0529

Table 3 The empirical power of the tests.

n	Power of the tests						
	φ	p1.	T_W	T_L	T_F	T_S	T_{IW}
25	0.25	0.25	0.4369	0.0049	0.0221	0.0967	0.2834
		0.50	0.8488	0.3716	0.4845	0.7321	0.7838
		0.75	0.9894	0.9564	0.9579	0.9887	0.9806
	0.75	0.25	0.1904	0.0138	0.0155	0.0417	0.1073
		0.50	0.2572	0.0859	0.1057	0.2308	0.2022
		0.75	0.4938	0.3341	0.3952	0.4249	0.3609
	1.25	0.25	0.0696	0.0518	0.0463	0.0363	0.0352
		0.50	0.0160	0.0700	0.2341	0.0399	0.0296
		0.75	0.0004	0.0015	0.0983	0.0016	0.0015
	1.75	0.25	0.0241	0.1294	0.1421	0.0832	0.0145
		0.50	0.0215	0.2474	0.5313	0.2032	0.1842
		0.75	0.1420	0.0738	0.2935	0.1056	0.0457
50	0.25	0.25	0.6210	0.0261	0.1138	0.2624	0.5699
		0.50	0.9652	0.8589	0.8596	0.9333	0.9559
		0.75	1.0000	0.9996	0.9995	1.0000	1.0000
	0.75	0.25	0.2002	0.0185	0.0300	0.0659	0.1681
		0.50	0.3111	0.1712	0.1355	0.2677	0.2861
		0.75	0.6317	0.5035	0.5064	0.5853	0.5223
	1.25	0.25	0.0472	0.0792	0.0844	0.0517	0.0402
		0.50	0.0715	0.1416	0.2488	0.0751	0.0759
		0.75	0.3589	0.4524	0.7095	0.4102	0.5535
	1.75	0.25	0.0479	0.2555	0.2838	0.1667	0.0574
		0.50	0.8012	0.8749	0.9287	0.8273	0.8471
		0.75	0.1147	0.0823	0.1673	0.1149	0.0623

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