ความสัมพันธ์ของกรีนบนโมนอยด์ของโคไฮเพอร์ซับสติติวชันเชิงเส้นชนิด $\tau = (n)$
Green's Relations on the Monoid of Linear Cohypersubstitutions of Type $\tau = (n)$

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บทคัดย่อ

โคไฮเพอร์ซับสติติวชันเชิงเส้นชนิด $\tau = (n)$ เป็นการส่งสัญลักษณ์การดำเนินการร่วมแบบ *n-ary* ไปยังพจน์ร่วมเชิงเส้นชนิด τ. สำหรับทุกโคไฮเพอร์ซับสติติวชันเชิงเส้น σ ชนิด $\tau = (n)$ ทำให้เกิดการส่ง $\hat{S}$ บนเซตของพจน์ร่วมเชิงเส้นชนิด τ. ทั้งหมด เซตของโคไฮเพอร์ซับสติติวชันเชิงเส้นชนิด τ ทั้งหมด ภายใต้การดำเนินการทวิภาค o_{coh} ซึ่งถูกกำหนดนิยามโดย $\sigma_1 o_{coh} \sigma_2 := \hat{\sigma}_1 o \sigma_2$ สำหรับทุก $\sigma_1,\sigma_2 \in Cohyp^{lin}(n)$ เป็นโมนอยด์ ในนี้เราจำแนกลักษณะความสัมพันธ์ของกรีนบน $Cohyp^{lin}(n)$.

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Abstract

Linear cohypersubstitutions of type $\tau = (n)$ are mappings which map the *n-ary* co-operation symbols to linear coterms of type τ. Every linear cohypersubstitution σ of type $\tau = (n)$ induces a mapping $\hat{S}$ on the set of all linear coterms of type τ. The set of all linear cohypersubstitutions of type τ under the binary operation o_{coh} which is defined by $\sigma_1 o_{coh} \sigma_2 := \hat{\sigma}_1 o \sigma_2$ for all $\sigma_1,\sigma_2 \in Cohyp^{lin}(n)$ forms a monoid. In this paper, we characterize Green's relations on $Cohyp^{lin}(n)$.

Keywords: linear cohypersubstitutions, linear coterms, superposition, Green's relations.

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Introduction

Let A be a non-empty set and n be a positive integer. The n-th copower $A^{\cup n}$ is the Cartesian product $A^{\cup n} := n \times A$, where $\underline{n}:=1,\ldots,n$. An element (i,a) in the copower corresponds to the element a in the i-th copy of A, for $1 \le i \le n$. A co-operation on A is a mapping $f^A:A\longrightarrow A^{\cup n}$ for some $n \ge 1$; the natural number n is called the arity of the co-operation f^A. We also need to recall that any n-ary co-operation f^A on set A can be uniquely expressed as a pair (f_1^A, f_2^A) of mappings, $f_1^A:A\longrightarrow \underline{n}$ and $f_2^A:A\longrightarrow A$; the first mapping gives the labeling used by f^A in mapping elements to copies of A, and the second mapping tells us what element of A is mapped to.

We shall denote by $cO_A^{(n)} = \{f^A \mid A\longrightarrow A^{\cup n}\}$ the set of all n-ary co-operations defined on A, and by $cO_A:= \cup_{n\ge 1} cO_A^{(n)}$ he set of all finitary co-operations defined on A. An indexed coalgebra is a pair $(A ; (f_i^A)_{i\in l})$, where f_i^A is $\underline{a}$ n_i-ary co-operation defined on A, and $\tau = (n_i)_{i\in l}$ for $n_i \ge 1$ is called the type of the coalgebra. Coalgebras were studied by Drbohlav[1]. In[2], the following superposition of co-operations was introduced. If $f^A\in cO_A^{(n)}$ and $g_0^A,\ldots,g^A_{n-1}\in cO_A^{(k)}$ then the k-ary co-operation $f^A[g_0^A,\ldots,g^A_{n-1}]:A\longrightarrow A^{\cup k}$ is defined by $a\longmapsto ((g^A_{f_1^A(a)})_1(f_2^A(a)),(g^A_{f_1^A(a)})_2(f_2^A(a)))$ for all $a\in A$. The co-operation $f^A[g_0^A,\ldots,g^A_{n-1}]$ is called the superposition of f^A and $g_0^A,\ldots,g^A_{n-1}$. It will also be denoted by $comp_k^n(f^A, g_0^A,\ldots,g^A_{n-1})$.

The injection co-operations $i_i^{n,A} : A\longrightarrow A^{\cup n}$ are special co-operations which are defined for each $0 \le i \le n-1$ by $i_i^{n,A} : A\longrightarrow A^{\cup n}$ with $a\longmapsto (i,a)$ for all $a\in A$. Then we get a multi-based algebra $((cO_A^{(n)})_{n\ge 1},(comp_k^n)_{k,n\ge 1},(i_i^{n,A})_{0\le i\le n-1})$, called the clone of co-operations on A. In[2], it is mentioned that this algebra is a clone, i.e. it satisfies the three clone axioms. In[3], K. Denecke and K. Saengsura gave a full proof of this fact and introduced the following coterms of type $\tau = (n_i)_{i\in l}$ were introduced. Let $(f_i)_{i\in l}$ be an indexed set of co-operation symbols such that for each $i\in l$. We say that symbol f_i has arity n_i, for $i\in l$. Let $U\{e_i^n \mid n \ge 1, n\in N, 0 \le j \le n-1\}$ be a set of symbols which is disjoint from the set $\{fi \mid i\in l\}$. We assign to each e_j^n the positive integer n as its arity. Then coterms of type τ are defined as follows:

(i) For every $i\in l$, the co-operation symbol f_i is an n_i-ary coterm of type τ.

(i) For every $n \ge 1$ and $0 \le j \le n-1$, the symbol e_j^n is an n-ary coterm of type τ.

(i) If $t_1,\ldots,t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1,\ldots,t_{n_i}]$ is an n-ary coterm of type τ and if $t_0,\ldots,t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0,\ldots,t_{n-1}]$ is an m-ary coterm of type τ, for every $i\in l$ and $n \ge 1$ and $0 \le j \le n-1$.

Let $cT_t^{(n)}$ be the set of all n-ary coterms of type τ and let $cT_t := \bigcup_{n\ge 1} cT_t^{(n)}$ be the set of all (finitary) coterms of type τ.

Definition 1.1 Let $t\in cT_t$ be a coterm and $E(t) = \{e_i^n \mid e_i^n$ occurs in t and $0 \le i \le n-1$. Then t is a linear coterm if for each $e_i^n\in E(t)$, e_i^n occurs only once in t.

We denote by $cT_t^{lin,(n)}$ the set of all n-ary linear coterms of type τ and $cT_t^{lin}:=\bigcup_{n\ge 1} cT_t^{lin,(n)}$ the set of all (finitary) linear coterms of type τ.

We define a family of superposition operations $(\bar{S}_m^n)_{m,n\ge 1}$ on this sequence, as follows.

Definition 1.2 The operation $\bar{S}_m^n:cT_t^{lin,(n)} \times (cT_t^{lin,(m)})^n\longrightarrow cT_t^{lin,(m)}$ is defined by induction on the complexity of linear coterm definition, as follows:

(i) If e_i^n is an n-ary linear coterm of type τ, $t_0,\ldots,t_{n-1}$ are m-ary linear coterms of type τ for $0 \le j \le n-1$ and $E(t_j)\cap E(t_k)=\varnothing$ for $j,k\in\{0,\ldots,n-1\}$ and $j\ne k$, then $\bar{S}_m^n(e_i^n,t_0,\ldots,t_{n-1}) := t_i$ is an m-ary linear coterm of type τ.

(ii) If f is an n-ary linear coterm of type τ, $t_1,\ldots,t_n$ are m-ary linear coterms of type τ and $E(t_j)\cap E(t_k)=\varnothing$ for $j,k\in\{1,\ldots,n\}$, then $\bar{S}_m^n(f,t_t,\ldots,t_n) := f[t_t,\ldots,t_n]$ is an n-ary linear coterm of type τ.

(iii) If f is an n-ary co-operation symbol, $S_1,\ldots,S_n$ are n-ary linear coterms of type τ where $E(s_j)\cap E(s_k)=\varnothing$ for $j,k\in\{1,\ldots,n\}$ and $t_1,\ldots,t_n$ are m-ary linear coterms of type τ where $E(t_j)\cap E(t_k)=\varnothing$ for $j,k\in\{1,\ldots,n\}$, then $\bar{S}_m^n(f[s_1,\ldots,s_n],t_t,\ldots,t_n):= f[\bar{S}_m^n(s_1,t_1,\ldots,t_n),\ldots, \bar{S}_m^n(s_n,t_1,\ldots,t_n)]$ is an n-ary linear coterm of type t.

Together with these operations we obtain a heterogeneous algebra $cT_t^{lin}:=((cT_t^{lin,(n)})_{n\geq 1}, (\bar{S}_m^{\,n})_{m,n\geq 1}, (e_j^n)_{0\leq j\leq n-1})$.

Definition 1.3 A linear cohypersubstitution of type t is a mapping $S : \{ f \}\to cT_t^{lin}$ from the set of all co-operation symbols to the set of all linear coterms which is inductively defined by the following steps:

(i) $\hat{\sigma}[e_j^n]:= e_j^n$ for every $n\geq 1$ and $0\leq j\leq n-1$,

(ii) $\hat{\sigma}[f]:= \sigma[f]$,

(iii) $\hat{\sigma}[f[t_1,...,t_n]]:= \bar{S}_n^{\,n}(\sigma(f),\hat{\sigma}[t_1],...,\hat{\sigma}[t_n])$ and assume that $\hat{\sigma}[t_j]$ is already defined and $E(t_j)$ are distinct for all $1\leq j\leq n$.

Let $Cohyp^{lin}(\tau)$ be the set of all linear cohypersubstitutions of type τ. Since the extension of a linear cohypersubstitution of type τ maps cT_τ^{lin} to cT_τ^{lin}, we may define a binary operation o_{coh} by $\hat{\sigma}_1\, o_{coh}\, \sigma_2:=\hat{\sigma}_1\, o\, \sigma_2$ where o is the usual composition of mappings. Let σ_{id} be the linear cohypersubstitution defined by $\sigma_{id}(f):=f$.

In 2016, D. Boonchari and K. Saengsura studied the monoid of cohypersubstitutions of type $\tau=(n)$[4]. In this paper, we characterize Green's relations on $Cohyp^{lin}(n)$.

Main results

In this section, we obtain the linear cohypersubstitutions σ_t and σ_s which are R-related, L-related, H-related, D-related and J-related as following theorem:

We characterize the Green's relation R on $Cohyp^{lin}(n)$ and we recall the definition of Green's relation R i.e., let a, b be elements of semigroup S. Then $a\,R\,b$ if and only if there exists x,y in S such that $xa=b, yb=a$.

Theorem 2.1 Let $\sigma_t, \sigma_s\in Cohyp^{lin}(n)$. If $t = e_i^n$, $s = e_j^n\in cT_t^{lin,(n)}$ for all $i,j\in\{0,...,n-1\}$ then $\sigma_t\, R\, \sigma_s$.

Proof Assume that $t = e_i^n$, $s = e_j^n\in cT_t^{lin,(n)}$ for all $i,j\in\{0,...,n-1\}$. We will show that there are σ_r, $\sigma_w\in Cohyp^{lin}(n)$ such that $\sigma_t = \sigma_s\, o_{coh}\, \sigma_r$ and $\sigma_s = \sigma_t\, o_{coh}\, \sigma_w$.

Since $\sigma_s(f) = s = e_j^n$ and $\hat{\sigma}_t[e_j^n]=e_j^n$, then $\sigma_s(f) = e_j^n$

$= \hat{\sigma}_t[e_j^n]$

$= \hat{\sigma}_t[\sigma_{e_j^n}(f)]$

$= \hat{\sigma}_t[\sigma_s(f)]$

$= (\sigma_t\, o_{coh}\, \sigma_s)(f)$.

Therefore, $\sigma_s = \sigma_t\, o_{coh}\, \sigma_s$.

Similarly, one can show that $\sigma_t = \sigma_s\, o_{coh}\, \sigma_r$ for some $\sigma_r\in Cohyp^{lin}(n)$.

This implies that $\sigma_t\, R\, \sigma_s$.

Theorem 2.2 Let $\sigma_t,\sigma_s\in Cohyp^{lin}(n)$. If $t = f[e^n_{jn_i}, ..., e^n_{jn_i}]\in cT_t^{lin,(n)}$ and $s = f[e^n_{j_0},..., e^n_{j_{n-1}}]\in cT_t^{lin,(n)}$ where $i_o,...,i_{n-1}, j_o,...,j_{n-1}\in\{0,...,j_{n-1}\}$ then $\sigma_t\, R\, \sigma_s$.

Proof Let $r = f[r_1,...,r_n]\in cT_t^{lin,(n)}$ such that $r_{j_k} = e^n_{i_k}$ for all $j_k\in\{0,...,n-1\}$ and $k = 0,...,n-1$.

Then $\sigma_t(f) = f[e^n_{j_0},..., e^n_{j_{n-1}}]$, and so $(\sigma_s\, o_{coh}\, \sigma_r)(f) = [\hat{\sigma}_r(f)]$

$= \hat{\sigma}_s[f[r_1,...,r_n]]$

$= \sigma_s(f)[r_1,...,r_n]$

$= (f[e^n_{j_0},..., e^n_{j_{n-1}}][r_1,...,r_n]$

$= f[e^n_{j_0}[r_1,...,r_n],..., e^n_{j_{n-1}}[r_1,...,r_n]]$

$= f[e^n_{i_0},..., e^n_{i_{n-1}}]$

$= t$

$= \sigma_t(f)$.

Therefore, $\sigma_s\, o_{coh}\, \sigma_r = \sigma_t$.

Similarly, one can show that $\sigma_s = \sigma_t\, o_{coh}\, \sigma_w$ for some $\sigma_w\in Cohyp^{lin}(n)$.

Hence, $\sigma_t\, R\, \sigma_s$.

Therefore, $(\sigma_t, \sigma_s)\in R$.

For linear cohypersubstitutions σ_t, σ_s such that t and s are different form i.e., $t\in\{e_i^n \mid 0\leq i\leq n-1\}$ and $s\in cT_t^{lin,(n)}\backslash\{e_i^n \mid 0\leq i\leq n-1\}$, we have that $(\sigma_t, \sigma_s)\notin R$ as the following example:

Example 2.3 Let $\sigma_t, \sigma_s\in Cohyp^{lin}(n)$ and $t = e^n_i$, $s = f[e^n_{j_0},...,e^n_{j_{n-1}}]\in cT_t^{lin,(n)}$ for all $i, J_o,..., J_{n-1}\in\{0,..., n-1\}$ and $E(s)$ be distinct.

Assume that $(\sigma_t, \sigma_s)\in R$.

Then there is $\sigma_w\in Cohyp^{lin}(n)$. such that $\sigma_s = \sigma_t\, o_{coh}\, \sigma_w$.

Hence

$f[e^n_{j_0},...,e^n_{j_{n-1}}] = s$

$= \sigma_s(f)$

$= \hat{\sigma}_t[\sigma_w(f)]$

$= \hat{\sigma}_t[w]$.

But we cannot find $w \in cT_t^{lin,(n)}$ such that

$\hat{\sigma}_t[w] = f[e^n_{j_1},...,e^n_{j_n}]$.

So $(\sigma_t, \sigma_s) \in R$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t R \sigma_s$ is $n^2 + (n!)^2$.

Next, we characterize the Green's relation L on $Cohyp^{lin}(n)$ and we recall the definition of Green's relation L i.e., $a L b$ if and only if there exists u, v in S such that $au = v, bv = u$.

Theorem 2.4 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t, s \in \{e^n_i \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. If $\sigma_t L \sigma_s$, then $t = s$.

Proof Assume that $\sigma_t L \sigma_s$.

Then there are $\sigma_u, \sigma_v \in Cohyp^{lin}(n)$ such that $\sigma_t = \sigma_u \, o_{coh} \, \sigma_s$ and $\sigma_s = \sigma_v \, o_{coh} \, \sigma_t$.

Let $\sigma_t(f) = t = e^n_j$ and $\sigma_s(f) = s = e^n_j$.

Then

$e^n_i = t$

$= \sigma_t(f)$

$= \hat{\sigma}_u[\sigma_s(f)]$

$= \hat{\sigma}_u[e^n_i]$

$= e^n_i$

$= s$.

Therefore, $t = s$.

For linear cohypersubstitutions σ_t, σ_s such that $t, s \in \{e^n_i \mid 0 \leq i \leq n\text{-}1\}$. and $t \neq s$, we have that $(\sigma_t, \sigma_s) \in L$. as the following example:

Example 2.5 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$

Assume that $t = e_i^n$, $s = e_j^n \in cT_t^{lin,(n)}$ for all $i,j \in \{0,...,n\text{-}1\}$ and $i \neq j$.

Then $e^n_i = t = \sigma_t(f)$ and $e^n_j = s = \sigma_s(f)$.

Since for all $\sigma_s \in Cohyp^{lin}(n)$, we have that $\hat{\sigma}_u[e^n_j] = e^n_j$. Then $\hat{\sigma}_u[\sigma_s(f)] = \sigma_s(f) \neq \sigma_t(f)$.

Therefore, $(\sigma_t, \sigma_s) \in L$.

Theorem 2.6 If $t = f[e^n_{i_o}, ..., e^n_{i_{n-1}}] \in cT_t^{lin,(n)}$ and $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ where $i_o,...,i_{n-1}, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$, then $\sigma_t L \sigma_s$.

Proof Let $v = f[v_1,...,v_n] \in cT_t^{lin,(n)}$ such that $v_1,...,v_n \in \{e^n_i \mid i = 0,...,n\text{-}1\}$ and $v_1[e^n_{i_0},...,e^n_{i_{n-1}}] = e^n_{j_0},...,v_n[e^n_{i_o},...,e^n_{i_{n-1}}] = e^n_{j_{n-1}}$.

Then

$\hat{\sigma}_v[\sigma_t(f)] = \hat{\sigma}_v[f[e^n_{i_0},...,e^n_{i_{n-1}}]]$

$= \sigma_v(f)[e^n_{i_0},...,e^n_{i_{n-1}}]$

$= (f[v_1,...,v_n])[e^n_{i_0},...,e^n_{i_{n-1}}]$

$= f[v_1[e^n_{i_0},...,e^n_{i_{n-1}}] = e^n_{j_0},...,v_n[e^n_{i_0},...,e^n_{i_{n-1}}]]$

$= f[e^n_{j_0},...,e^n_{j_{n-1}}]$

$= s$

$= \sigma_s(f)$.

Therefore, $\sigma_v \, o_{coh} \, \sigma_t = \sigma_s$.

Similarly, one can show that $\sigma_t = \sigma_u \, o_{coh} \, \sigma_s$ for some $\sigma_u \in Cohyp^{lin}(n)$.

Hence, $\sigma_t L \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t L \sigma_s$ is $n + (n!)^2$.

Next, we characterize the Green's relation H on $Cohyp^{lin}(n)$.

Theorem 2.7 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t, s \in \{e^n_i \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t H \sigma_s$ if and only if $t = s$.

Proof Assume that $\sigma_t H \sigma_s$.

Then $\sigma_t H \sigma_s$ and $\sigma_t R \sigma_s$.

By Theorem 2.4, we get that $t = s$.

Similarly, assume that $t = s$.

Then $\sigma_t = \sigma_s$.

Since and are equivalence relations,

we have $\sigma_t L \sigma_s$ and $\sigma_t R \sigma_s$.

Therefore, $\sigma_t H \sigma_s$.

Theorem 2.8 Let $t, s \in cT_t^{lin,(n)} \setminus \{e^n_i \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t H \sigma_s$.

Proof Let $t = f[e^n_{i_o}, ..., e^n_{i_{n-1}}] \in cT_t^{lin,(n)}$ and $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ for $i_o,...,i_{n-1}, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$.

By Theorem 2.2, we have that $\sigma_t R \sigma_s$.

By Theorem 2.6, we have that $\sigma_t L \sigma_s$.

Therefore, $\sigma_t H \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t H \sigma_s$ is $n + (n!)^2$.

Next, we characterize the Green's relation D on $Cohyp^{lin}(n)$.

Theorem 2.9 Let $(\sigma_t, \sigma_s) \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t D \sigma_s$.

Proof Since $\sigma_t L \sigma_t$ and by Theorem 2.2,

we have that $\sigma_t R \sigma_t$.

Then $\sigma_t D \sigma_t$.

Theorem 2.10 Let $t, s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t D \sigma_s$.

Proof Let $t = f[e^n_{i_o}, ..., e^n_{i_{n-1}}] \in cT_t^{lin,(n)}$ and $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ for $i_o,...,i_{n-1}, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$.

By Theorem 2.2, we have that $\sigma_t R \sigma_s$.

By Theorem 2.6, we get that $\sigma_t L \sigma_s$.

Therefore, $\sigma_t D \sigma_s$.

For linear cohypersubstitutions σ_t, σ_s such that t and s are different form i.e., $t \in \{e_i^n \mid 0 \leq i \leq n\text{-}1\}$ and $s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid 0 \leq i \leq n\text{-}1\}$, we have that $(\sigma_t, \sigma_s) \notin D$ as the following example:

Example 2.11 Let $\sigma_t, \sigma_s \in Cohyp^{lin}(n)$ and $t = e_i^n$, $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ for all $i, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$ and $E(s)$ be distinct.

Then $\sigma_t(f) = e_i^n$ and $\sigma_s(f) = f[e^n_{j_0},...,e^n_{j_{n-1}}]$.

By Theorem 2.4, we get that $\sigma_t L \sigma_s$.

But by Theorem 2.3, we have that $(\sigma_t, \sigma_s) \notin R$.

Hence, $(\sigma_t, \sigma_s) \notin D$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t D \sigma_s$ is $n^2 + (n!)^2$.

Next, we characterize the Green's relation J on $Cohyp^{lin}(n)$.

Theorem 2.12 Let $(\sigma_t, \sigma_s) \in Cohyp^{lin}(n)$ and $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t J \sigma_s$.

Proof Let $t = e_i^n$, $s = e_j^n$ and $u \in cT_t^{lin,(n)}$.

Since $\hat{\sigma}_u[e_k^n] = e_k^n$ for all $k = 0,...,n\text{-}1$.

we have

$\sigma_t(f) = e_i^n$

$= \hat{\sigma}_s[e_i^n]$

$= \hat{\sigma}_u[\hat{\sigma}_s[e_i^n]]$

$= \hat{\sigma}_u[\hat{\sigma}_s[\sigma_{e_i^n}(f)]]$

$= \hat{\sigma}_u[\hat{\sigma}_s[\sigma_t(f)]]$.

Therefore, $\sigma_t = \sigma_u \, o_{coh} \, \sigma_s \, o_{coh} \, \sigma_t$.

Similarly, one can show that $\sigma_s = \sigma_x \, o_{coh} \, \sigma_t \, o_{coh} \, \sigma_y$ for some $\sigma_x, \sigma_y \in Cohyp^{lin}(n)$.

Hence, $\sigma_t J \sigma_s$.

Theorem 2.13 Let $t, s \in cT_t^{lin,(n)} \setminus \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$. Then $\sigma_t J \sigma_s$.

Proof Let $t = f[e^n_{i_o}, ..., e^n_{i_{n-1}}] \in cT_t^{lin,(n)}$ and $s = f[e^n_{j_o}, ..., e^n_{j_{n-1}}] \in cT_t^{lin,(n)}$ for $i_o,...,i_{n-1}, j_o,...,j_{n-1} \in \{0,...,n\text{-}1\}$.

We let $r = f[r_1,...,r_n]$ such that $r_{j_k} = e^n_{j_k}$ where $j_k \in \{0,...,n\text{-}1\}$ and $k = 0,...,n\text{-}1$.

By Theorem 2.2, we get that $\sigma_t(f)] = \hat{\sigma}_s[\sigma_r(f)]$.

Let $v = (f)[e^n_o,...,e^n_{n-1}] \in cT_t^{lin,(n)}$.

Then

$\hat{\sigma}_v[\sigma_t(f)] = \hat{\sigma}_v[f[e^n_{i_0},...,e^n_{i_{n-1}}]]$

$= \sigma_v(f)[e^n_{i_0},...,e^n_{i_{n-1}}]$

$= (f[v_1,...,v_n])[e^n_{i_0},...,e^n_{i_{n-1}}]$

$= f[v_1[e^n_{i_0},...,e^n_{i_{n-1}}] = e^n_{j_0},...,v_n[e^n_{i_0},...,e^n_{i_{n-1}}]]$

$= f[e^n_{j_0},...,e^n_{j_{n-1}}]$

$= t$

$= \sigma_t(f)$.

Therefore, $\sigma_v \, o_{coh} \, \sigma_s \, o_{coh} \, \sigma_r = \sigma_t$.

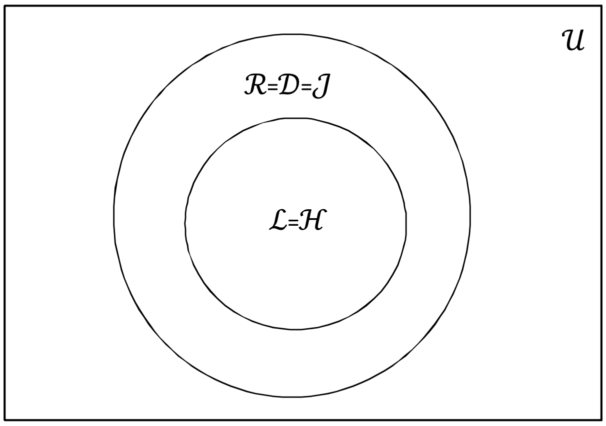
Similarly, one can show that $\sigma_s = \sigma_x \, o_{coh} \, \sigma_t \, o_{coh} \, \sigma_y$ for some $\sigma_x, \sigma_y \in Cohyp^{lin}(n)$.

Hence, $\sigma_t J \sigma_s$.

Remark The number of pairs (σ_t, σ_s) in which $\sigma_t J \sigma_s$ is $n^2 + (n!)^2$.

We conclude the R, L, H, D and J as the following diagram:

$U = \{(\sigma_t, \sigma_s) \mid t, s \in cT_t^{lin,(n)}\}$.



If $t, s \in \{e_i^n \mid n \geq 1, 0 \leq i \leq n\text{-}1\}$ and $t = s$ in L, then $L \subseteq R$.

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