ตัวกรองไฮเพอร์วิภัชนัยแบบอ่อนของพีชคณิตบีอีไฮเพอร์
Fuzzy Weak Hyper Filters of Hyper BE-Algebras

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Received: 5 October 2019 ; Revised: 3 March 2020 ; Accepted: 24 March 2020

บทคัดย่อ

ในบทความวิจัยนี้ได้แนะนำแนวคิดของตัวกรองไฮเพอร์วิภัชนัยแบบอ่อนในพีชคณิตบีอีไฮเพอร์ และได้ศึกษาสมบัติบางประการของตัวกรองไฮเพอร์วิภัชนัยแบบอ่อน จากนั้นได้แสดงว่าเซตของตัวกรองไฮเพอร์วิภัชนัยแบบอ่อนทั้งหมดของพีชคณิตบีอีไฮเพอร์เป็นแลตทิซบริบูรณ์ที่มีการแจงแจง ยิ่งไปกว่านั้นได้จำแนกลักษณะเฉพาะของพีชคณิตบีอีไฮเพอร์นอเทอร์เรียน และพีชคณิตบีอีไฮเพอร์อาร์ทิเนียน โดยใช้ตัวกรองไฮเพอร์วิภัชนัยแบบอ่อน

คำสำคัญ: ตัวกรองไฮเพอร์วิภัชนัย ตัวกรองไฮเพอร์วิภัชนัยแบบอ่อน พีชคณิตบีอี พีชคณิตบีอีไฮเพอร์

Abstract

The aim of this work is to introduce the notion of fuzzy weak hyper filters in hyper BE-algebras and investigate some of their properties. This research shows that the set of all fuzzy weak hyper filters of hyper BE-algebras is a distributive complete lattice. Also, the concepts of Noetherian hyper BE-algebras and Artinian hyper BE-algebras are characterized by their fuzzy weak hyper filters.

Keywords: fuzzy hyper filter, fuzzy weak hyper filter, BE-algebra, hyper BE-algebra

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Introduction

The fuzzy set was introduced by Zadeh[1] as a function from a nonempty set X to the unit interval $[0,1]$. Later, many researchers have discussed the generalizations of the concepts of fuzzy sets with applications in computing, logic and many ramifications of pure and applied mathematics. Kim and Kim[2] introduced the notion of BE-algebras, as a generalization of BCK-algebras[3] and BCI-algebras[4]. In 2010, the concept of fuzzy ideals in BE-algebras was introduced and some of its properties were investigated by Song, Jun and Lee[5]. Then, Dymek and Walendziak[6] studied and characterized the concept of fuzzy filters in BE-algebras.

The hyperstructure theory was introduced by Marty[7] in 1934 as a generalization of ordinary algebraic structures. Radfar, Rezaei and Borumand Saeid[8] applied the hyper theory to introduce the notion of hyper BE-algebras, as a generalization of BE-algebras. In 2015, Cheng and Xin[9] investigated some types of hyper filters on hyper BE-algebras.

In this work, the concept of fuzzy weak hyper filters of hyper BE-algebras is introduced, and its properties are considered. Finally, the concepts of Noetherian hyper BE-algebras and Artinian hyper BE-algebras are characterized by their fuzzy weak hyper filters.

Preliminaries

Let X be a nonempty set. The mapping $\circ$, $X \times X \to P^*(X)$, where $P^*(X)$ denotes the set of all nonempty subsets of H, is called a *hyperoperation*[10-13] on H. The hyperstructure $(H,\circ)$ is called a *hypergroupoid.* Let A and B be any two nonempty subsets of H and $x \in H$. Then, we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b,$$

$A \circ x = A \circ \{x\}$ and $x \circ B = \{x\} \circ B$.

Let H be a nonempty set and $\circ: X \times X \to P^*(X)$ be a hyperoperation. Then $(H,\circ,1)$ is called a *hyper BE-algebra*[8] if it satisfies the following axioms:

(i) $x < 1$ and $x < x$;

(ii) $x \circ (y \circ z) = y \circ (x \circ z)$;

(iii) $x \in 1 \circ x$;

(iv) $1 < x$ implies $x = 1$;

for all $x, y, z \in H$, where the relation "$<$" is defined by $x<y$ if and only if $1 \in x \circ y$.

Example 2.1[8] Define the hyperoperation "$\circ$" on $\mathbb{R}$ as follows:

$$x \circ y = \begin{cases} \{y\} & \text{if } \quad x = 1; \\ \mathbb{R} & \text{otherwise.} \end{cases}$$

Then, $(\mathbb{R},\circ,1)$ is a hyper BE-algebra.

Example 2.2[8] Let $X = \{1,a,b\}$. Define the hyperoperation "$\circ$" on as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1,a\}$	$\{1,a,b\}$	$\{1,a\}$
b	$\{1,a,b\}$	$\{a\}$	$\{1,a,b\}$

Then, $(H,\circ,1)$ is a hyper BE-algebra.

Let F be a nonempty subset of a hyper BE-algebra H and $1 \in F$. Then F is called:

(i) a *weak hyper filter*[8] of H if $x \circ y \subseteq F$ and $x \in F$, then $y \in F$, for all $x,y \in F$;

(ii) a *hyper filter*[8] of H if $x \circ y \approx F$ and $x \in F$, then $y \in F$, where $x \circ y \approx F$ means that $x \circ y \cap F \neq \varnothing$, for all $x,y \in H$.

Note that every hyper filter of a hyper BE-algebra H is a weak hyper filter of H, but the converse is not true in general[8]. In this paper, we will focus on weak hyper filters of hyper BE-algebras.

Lemma 2.3 If $\{F_i : i \in I\}$ is a chain of a family of weak hyper filters of a hyper BE-algebra H, then $\bigcup_{i \in I} F_i$ is also a weak hyper filter of H.

Proof. Let $\bigcup_{i \in I} F_i$. Clearly, $1 \in F$. Let $x,y \in H$ such that $x \circ y \subseteq F$ and $x \in F$. Then $x \circ y \subseteq F_i$ and $x \in F_i$ for some $i,j \in I$. Assume that $F_i \subseteq F_j$. It follows that $x \circ y \subseteq F_j$ and $x \in F_j$. Since F_j is a weak hyper filter of H, we have $y \in F_j \subseteq F$. Hence, F is a weak hyper filter of H.

A *fuzzy set*[1] of a nonempty set X is a mapping $\mu: X \to [0,1]$. Then, the set $U(\mu;\alpha)=\{x \in X : \mu(x) \geq \alpha\}$ is called a *level subset* of μ. where $\alpha \in [0,1]$. Let μ and ν be any two fuzzy sets of a nonempty set X. Then $\mu \subseteq \nu$, means that $\mu(x) \leq v(x)$, for all $x \in X$. In addition, the intersection and the union of μ and v, denoted by $\mu \cap v$ and $\mu \cup v$, respectively, are defined by letting $x \in X$, $(\mu \cap v)(x) = \min\{\mu(x), v(x)\}$ and $(\mu \cup v)(x) = \max\{\mu(x), v(x)\}$.

Results

In this section, we introduce the notion of fuzzy weak hyper filters of hyper BE-algebras, and we investigate some fundamental properties of fuzzy weak hyper filters in hyper BE-algebras.

Definition 3.1 A fuzzy set μ of a hyper BE-algebra H is called a *fuzzy weak hyper filter* of H if it satisfies the following conditions:

(i) $\mu(1) \geq \mu(x)$;

(ii) $\mu(x) \geq \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}$;

for all $x, y \in H$.

Example 3.2 Let $H=\{1,a,b\}$ be a set with a hyperoperartion "o" on defined as follows:

o	1	a	b
1	{1}	{a,b}	{b}
a	{1}	{1,a}	{1,b}
b	{1}	{1,a,b}	{1}

Then, is a hyper BE-algebra[8]. We define a fuzzy set μ of H by $\mu(a) \leq \mu(b) \leq \mu(1)$. By routine computations, we have that μ is a fuzzy weak hyper filter of H.

Theorem 3.3 Let be a fuzzy set of a hyper BE-algebra H. Then μ is a fuzzy weak hyper filter of H if and only if its nonempty level subset $U(\mu;\alpha)=\{x \in H: \mu(x) \geq \alpha\}$ is a weak hyper filter of for all $\alpha \in [0,1]$.

Proof. Assume that μ is a fuzzy weak hyper filter of H. Let $\alpha \in [0,1]$ such that $U(\mu;\alpha) \neq \varnothing$. Then there exists $x_0 \in U(\mu;\alpha)$ such that $\mu(x_0) \geq \alpha$. Since $\mu(1) \geq \mu(x_0)$, $1 \in U(\mu;\alpha)$. Let $x,y \in H$ such that $x \circ y \subseteq U(\mu;\alpha)$ and $x \in U(\mu;\alpha)$. Then $\mu(z) \geq \alpha$, for all $z \in x \circ y$. Thus, $\mu(y) \geq \min\{\inf_{z \in x,y} \mu(z), \mu(x)\} \geq \alpha$, that is, $y \in U(\mu;\alpha)$. Hence, $U(\mu;\alpha)$ is a weak hyper filter of H.

Conversely, suppose that $\mu(1) \geq \mu(x_0) = \beta$ for some $x_0 \in H$ and $\beta \in [0,1]$. Then $U(\mu;\beta) \neq \varnothing$, and so $U(\mu;\beta)$ is a weak hyper filter of H. It follows that $1 \in U(\mu;\beta)$, which implies that $\mu(1) \geq \beta$. This is a contradiction. Thus, $\mu(1) \geq \mu(x)$, for all $x \in H$. Suppose that $\mu(a) < \min\{\inf_{z \in b,a} \mu(z), \mu(b)\}$ for some $a,b \in H$. Letting $\alpha = \frac{1}{2}\left(\mu(a) + \min\left\{\inf_{z \in b \circ a} \mu(z), \mu(b)\right\}\right)$.

We have $\mu(a) < \alpha < \min\{\inf_{z \in b,a} \mu(z), \mu(b)\} \leq \inf_{z \in b,a} \mu(z)$ and $\alpha < \mu(b)$. Then $b \circ a \subseteq U(\mu;\alpha)$ and $b \in U(\mu;\alpha)$. Since is a weak hyper filter of H, we have $a \in U(\mu;\alpha)$, that is, $\mu(a) \geq \alpha$. This is a contradiction. We obtain that $\mu(a) \geq \min\{\inf_{z \in b,a} \mu(z), \mu(b)\}$ for all $a,b \in H$. Therefore, μ is a fuzzy weak hyper filter of H.

Corollary 3.4 If μ is a fuzzy weak hyper filter of a hyper BE-algebra H, then the set $H_a=\{x \in H: \mu(x) \geq \mu(a)\}$ is a weak hyper filter of H for all $a \in H$.

Corollary 3.5 If μ is a fuzzy weak hyper filter of a hyper BE-algebra H, then the set $H_\mu=\{x \in H: \mu(x) = \mu(1)\}$ is a weak hyper filter of H.

Theorem 3.6 Let $F_1 \subset F_2 \subset \cdots F_n \subset \ldots$ be a strictly ascending chain of weak hyper filters of a hyper BE-algebra H and $\{t_n\}$ be a strictly decreasing sequence in $[0,1]$. Let μ be a fuzzy set of H, defined by $\mu(x)=$

$$\begin{cases} 0 & \text{if} \quad x \notin F_n \quad \text{for each } n \in \mathbb{N}; \\ t_n & \text{if} \quad x \in F_n - F_{n-1} \quad \text{for } n = 1, 2, \ldots; \end{cases}$$

for all $x \in H$, where $F_0 = \varnothing$. Then μ is a fuzzy weak hyper filter of H.

Proof. Let $F = \bigcup_{n \in \mathbb{N}} F_n$. By Lemma 2.3, F is a weak hyper filter of H. Then $\mu(1) = t_1 \geq \mu(x)$, for all $x \in H$. Let $x,y \in H$. Thus, we can divide to be two cases, as follows.

Case 1: $x \notin F$. Then $y \circ x \notin F$ or $y \notin F$. There exists $a \in y \circ x$ such that $x \notin F$. Thus, $\mu(a)=0$ or $\mu(y)=0$. Hence, $\min\{\inf_{z \in y,x} \mu(z), \mu(y)\}$.

Case 2: $x \in F_n - F_{n-1}$ for some $n = 1,2,\ldots$. Then $y \circ x \not\subseteq F_{n-1}$ or $y \notin F$. Thus, there exists $a \in y \circ x$ such that $a \notin F_{n-1}$. We obtain that, $\inf_{z \in y,x} \mu(z) \leq t_n$ or $\mu(y) \leq t_n$. Therefore, $\min\left\{\inf_{z \in y \circ x} \mu(z), \mu(y)\right\} \leq t_n = \mu(x)$. Consequently, μ is a fuzzy weak hyper filter of H.

Let μ and v be fuzzy sets of a nonempty set X. The *cartesian product*[14] of μ and v is defined by $(\mu \times v)(x, y) = \min\{\inf_{z \in y,x} \mu(z), \mu(b)$, for all $x,y \in X$.

Theorem 3.7 Let H be a hyper BE-algebra. If μ and v are fuzzy weak hyper filters of H, then $\mu \times v$ is a fuzzy weak hyper filter of $H \times H$.

Proof. Assume that μ and v are fuzzy weak hyper filters of H. Let $(x,y) \in H \times H$. Then

$(\mu \times \nu)(1,1) = \min\{\mu(1), \nu(1)\} \geq \min\{\mu(x), \nu(y)\} = (\mu \times \nu)(x, y)$. Now, let $(x_1, y_2), (x_2, y_2) \in H \times H$. Then $(\mu, \nu)(x_1, y_1)$

$$\begin{aligned}
&= \min\{\mu(x_1), \nu(y_1)\} \\
&\geq \min\{\min\{\inf_{z_1 \in x_2 \circ x_1} \mu(z_1), \mu(x_2)\}, \\
&\qquad \min\{\inf_{z_2 \in y_2 \circ y_1} \nu(z_2), \nu(y_2)\}\} \\
&\geq \min\{\inf_{\substack{z_1 \in x_2 \circ x_1 \\ z_2 \in y_2 \circ y_1}} \{\min\{\mu(z_1), \nu(z_2)\}, \\
&\qquad \min\{\mu(x_2), \nu(y_2)\}\}\} \\
&\geq \min\{\inf_{(z_1, z_2) \in (x_2, y_2) \circ (x_1, y_1)} (\mu \times \nu)(z_1, z_2), \\
&\qquad (\mu \times \nu)(x_2, y_2)\}.
\end{aligned}$$

Therefore, $\mu \times \nu$ is a fuzzy weak hyper filter of $H \times H$.

Let be a fuzzy set of a nonempty set X, $\alpha \in [0, 1 - \sup_{x\in X} \mu(x)]$and $\beta \in [0,1]$. Then:

(i) the mapping $\mu^T_\alpha : X \to [0,1]$ is called a *fuzzy translation*[15] of μ if $\mu^T_\alpha(x) = \mu(x) + \alpha$, for all $x \in X$;

(ii) the mapping $\mu^M_\beta : X \to [0,1]$ is called a *fuzzy multiplication*[15] of μ if $\mu^M_\beta(x) = \beta\mu(x)$, for all $x \in X$;

(iii) the mapping $\mu^{MT}_{\beta,\alpha} : X \to [0,1]$ is called a *fuzzy magnified translation*[16] of μ if $\mu^{MT}_{\beta,\alpha}(x) = \beta\mu(x) + \alpha$, for all $x \in X$.

Theorem 3.8 Let H be a hyper BE-algebra, μ be a fuzzy set of H, $\alpha \in [0, 1 - \sup_{x\in H} \mu(x)]$ and $\beta \in [0,1]$. Suppose that $\mu^{MT}_{\beta,\alpha}$ is a fuzzy magnified translation of μ, with respect to α and β. Then μ is a fuzzy weak hyper filter of H if and only if $\mu^{MT}_{\beta,\alpha}$ is a fuzzy weak hyper filter of H.

Proof. Assume that μ is a fuzzy weak hyper filter of H. Let $a \in H$. Since $\mu(1) \geq \mu(a)$, we have $\mu^{MT}_{\beta,\alpha}(1) = \beta\mu(1) + \alpha \geq \beta\mu(a) + \alpha = \mu^{MT}_{\beta,\alpha}(a)$, for all $a \in H$. Let $x,y \in H$. Then

$$\begin{aligned}
\mu^{MT}_{\beta,\alpha}(x) &= \beta\mu(x) + \alpha \\
&\geq \beta \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} + \alpha \\
&= \min\{\inf_{z \in y \circ x} (\beta\mu(z) + \alpha), \beta\mu(y) + \alpha\} \\
&= \min\{\inf_{z \in y \circ x} \mu^{MT}_{\beta,\alpha}(z), \mu^{MT}_{\beta,\alpha}(y)\}.
\end{aligned}$$

Hence, $\mu^{MT}_{\beta,\alpha}$ is a fuzzy weak hyper filter of H.

Conversely, assume that $\mu^{MT}_{\beta,\alpha}$ is a fuzzy weak hyper filter of H. Let $x,y \in H$. Consider $\beta\mu(1) + \alpha = \mu^{MT}_{\beta,\alpha}(1) \geq \mu^{MT}_{\beta,\alpha}(x) = \beta\mu(x) + \alpha$ and

$$\begin{aligned}
\beta\mu(x) + \alpha &= \mu^{MT}_{\beta,\alpha}(x) \\
&\geq \min\{\inf_{z \in y \circ x} \mu^{MT}_{\beta,\alpha}(z), \mu^{MT}_{\beta,\alpha}(y)\} \\
&= \min\{\inf_{z \in y \circ x} (\beta\mu(z) + \alpha), \beta\mu(y) + \alpha\} \\
&= \min\{\beta(\inf_{z \in y \circ x} \mu(z)) + \alpha, \beta\mu(y) + \alpha\} \\
&= \beta\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} + \alpha.
\end{aligned}$$

Since $\beta > 0$ and $\alpha \geq 0$, we have $\mu(x) \geq \min\left\{\inf_{z \in y \circ x} \mu(z), \mu(y)\right\}$ and $\mu(1) \geq \mu(x)$, for all $x,y \in H$. Hence, μ is a fuzzy weak hyper filter of H.

Corollary 3.9 Let H be a hyper BE-algebra, μ be a fuzzy set of H, $\alpha \in [0, 1 - \sup_{x\in X} \mu(x)]$, and $\beta \in [0,1]$. Suppose that μ^T_α is a fuzzy translation and is a fuzzy multiplication of with respect to and , respectively. Then the following conditions are equivalent:

(i) μ is a fuzzy weak hyper filter of H ;

(ii) μ^T_α is a fuzzy weak hyper filter of H ;

(iii) μ^M_β is a fuzzy weak hyper filter of H.

Theorem 3.10 If μ and ν are fuzzy weak hyper filters of a hyper BE-algebra H, then $\mu \cap \nu$ is a fuzzy weak hyper filter of H.

Proof. Assume that μ and ν are fuzzy weak hyper filters of a hyper BE-algebra H. Let $x,y \in H$. Then

$$\begin{aligned}
(\mu \cap \nu)(1) &= \min\{\mu(1), \nu(1)\} \\
&\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x)
\end{aligned}$$

and

$$\begin{aligned}
(\mu \cap \nu)(x) &= \min\{\mu(x), \nu(x)\} \\
&\geq \min\{\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}, \\
&\qquad \min\{\inf_{z \in y \circ x} \nu(z), \nu(y)\}\} \\
&= \min\{\inf_{z \in y \circ x} \{\min\{\mu(z), \nu(z)\}\}, \\
&\qquad \min\{\mu(y), \nu(y)\}\} \\
&= \min\{\inf_{z \in y \circ x} (\mu \cap \nu)(z), (\mu \cap \nu)(y)\}.
\end{aligned}$$

Hence, $\mu \cap \nu$ is a fuzzy weak hyper filter of H.

Theorem 3.11 If μ and ν are fuzzy weak hyper filters of a hyper BE-algebra H such that $\mu \subseteq \nu$ or $\nu \subseteq \mu$, then $\mu \cup \nu$ is a fuzzy weak hyper filter of H.

Proof. Assume that μ and ν and are fuzzy weak hyper filters of a hyper BE-algebra H such that $\mu \subseteq \nu$ or $\nu \subseteq \mu$. Let $x,y \in H$. Then

$$(\mu \cup \nu)(1) = \max\{\mu(1), \nu(1)\} \geq \max\{\mu(x), \nu(x)\} = (\mu \cup \nu)(x).$$

Now,

$$\begin{aligned}(\mu \cup \nu)(x) &= \max\{\mu(x), \nu(x)\} \\ &\geq \max\{\min\{\inf_{z\in y\circ x} \mu(z), \mu(y)\}, \min\{\inf_{z\in y\circ x} \nu(z), \nu(y)\}\} \\ &= \min\{\max\{\inf_{z\in y\circ x} \mu(z), \mu(y)\}, \max\{\inf_{z\in y\circ x} \nu(z), \nu(y)\}\} \\ &= \min\{\inf_{z\in y\circ x} \{\max\{\mu(z), \nu(z)\}\}, \max\{\mu(y), \nu(y)\}\} \\ &= \min\{\inf_{z\in y\circ x} (\mu \cup \nu)(z), (\mu \cup \nu)(y)\}.\end{aligned}$$

In general, $\max\{\min\{\ \}\}\min\{\max\{\ \}\}$. Suppose for this case

$$\max\{\min\{\inf_{z\in y\circ x} \mu(z), \mu(y)\}, \min\{\inf_{z\in y\circ x} \nu(z), \nu(y)\}\} \neq \min\{\max\{\inf_{z\in y\circ x} \mu(z), \mu(y)\}, \max\{\inf_{z\in y\circ x} \nu(z), \nu(y)\}\}.$$

Then there exists $\alpha\in[0,1]$ such that

$$\max\{\min\{\inf_{z\in y\circ x} \mu(z), \mu(y)\}, \min\{\inf_{z\in y\circ x} \nu(z), \nu(y)\}\} < \alpha < \min\{\max\{\inf_{z\in y\circ x} \mu(z), \mu(y)\}, \max\{\inf_{z\in y\circ x} \nu(z), \nu(y)\}\}.$$

Thus, $\alpha < \min\left\{\inf_{z\in y\circ x} \mu(z), \mu(y)\right\}$. On the other hand, $\min\left\{\inf_{z\in y\circ x} \mu(z), \mu(y)\right\} < \alpha$, which is a contradiction. This completes the proof.

Then, we have the following corollary.

Corollary 3.12 Let $\{\mu_i : i\in\Lambda\}$ be a nonempty set of a family of fuzzy weak hyper filters of a hyper BE-algebra H, where Λ is an arbitrary indexed set. Then the following statements hold:

(i) $\bigcap_{i\in\Lambda} \mu_j$ is a fuzzy weak hyper filter of H;

(ii) if $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i,j\in\Lambda$, then $\bigcap_{i\in\Lambda} \mu_j$ is a fuzzy weak hyper filter of H.

Next, we denote by $FHF(H)$ the set of all fuzzy weak hyper filters of a hyper BE-algebra H. By Corollary 3.12, we obtain the following theorem.

Theorem 3.13 Let H be a hyper BE-algebra and $(FHF(H);\subseteq)$ be a totally ordered set by the set inclusion. Then $(FHF(H);\subseteq,\vee,\wedge)$ is a complete lattice, where

$$\wedge\{\mu_i \in FHF(H) : i\in\wedge\} = \bigcap_{i\in\Lambda} \mu_i,$$
$$\vee\{\mu_i \in FHF(H) : i\in\wedge\} = \bigcap_{i\in\Lambda} \mu_i.$$

Lemma 3.14 Let H be a hyper BE-algebra and $(FHF(H);\subseteq)$ be a totally ordered set. Then $\mu\cap(\nu\cup\lambda) = (\mu\cap\nu)\cup(\mu\cap\lambda)$ and $\mu\cup(\nu\cap\lambda) = (\mu\cup\nu)\cap(\mu\cup\lambda)$, for all $\mu,\nu,\lambda\in FHF(H)$.

Proof. Let $\mu,\nu,\lambda\in FHF(H)$ and $x\in H$. Then $(\mu\cap(\nu\cup\lambda))(x)$

$$\begin{aligned} &= min\{\mu(x), (\nu\cup\lambda)(x)\} \\ &= min\{\mu(x), max\{\nu(x), \lambda(x)\}\} \\ &= max\{min\{\mu(x),\nu(x)\}, min\{\mu(x),\lambda(x)\}\} \\ &= max\{(\mu\cap\nu)(x), (\mu\cap\lambda)(x)\} \\ &= ((\mu\cap\nu)\cup(\mu\cup\lambda))(x).\end{aligned}$$

Hence, $\mu\cap(\nu\cup\lambda) = (\mu\cap\nu)\cup(\mu\cap\lambda)$. Similarly, we can prove that $\mu\cup(\nu\cap\lambda) = (\mu\cup\nu)\cap(\mu\cup\lambda)$.

From Lemma 3.14, we have the following theorem.

Theorem 3.15 Let be a hyper BE-algebra and be a totally ordered set. Then is a distributive complete lattice.

Next, we characterize Noetherian hyper BE-algebras and Artinian hyper BE-algebras using their fuzzy weak hyper filters.

A hyper BE-algebra H is called *Noetherian* if H satisfies the ascending chain condition on weak hyper filters, that is, for any weak hyper filters $F_1, F_2, F_3, \ldots$ of H, with $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots \subseteq F_i \subseteq \ldots$.

There exists $n\in\mathbb{N}$ such that $F_i = F_i+1$ for all $i \geq n$.

A hyper BE-algebra H is called *Artinian* if H satisfies the descending chain condition on weak hyper filters, that is, for any weak hyper filters $F_1, F_2, F_3, \ldots$ of H, with $F_1 \subseteq F_2 \subseteq F_3 \subseteq \ldots \subseteq F_i \subseteq \ldots$.

There exists $n\in\mathbb{N}$ such that $F_i = F_i+1$ for all $i \geq n$.

Theorem 3.16 Let H be a hyper BE-algebra. Then H is Noetherian if and only if for every fuzzy weak hyper filter μ of H, the set $Im(\mu) = \{\mu(x): x\in H\}$ is a well-ordered subset of $[0,1]$.

Proof. Assume that H is Noetherian. Suppose that there exists a fuzzy weak hyper filter μ of H such that $\mathrm{Im}(\mu)$ is not a well-ordered subset of $[0,1]$. Then there exists a strictly infinite decreasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $\mu(x_n) = t_n$ for some $x_n \in H$. Let $I_n = U(\mu;t_n) = \{x\in H: \mu(x)\geq t_n\}$. By Theorem 3.3, I_n is a weak hyper filter of H, for all $n\in\mathbb{N}$. Moreover, $I_1 \subset I_2 \subset I_3 \subset \ldots$ is a strictly infinite ascending chain of weak hyper filters of H. This is a contradiction that H is Noetherian. Therefore, $Im(\mu)$ is a well-ordered subset of $[0,1]$, for each fuzzy weak hyper filter μ of H.

Conversely, assume that for every fuzzy weak hyper filter μ of H, the set $Im(\mu) = \{\mu(x): x\in H\}$ is a well-ordered subset of . Suppose that is not Noetherian. Then there exists a strictly infinite ascending chain $F_1 \subset F_2 \subset F_3 \subset \ldots \subset F_n \subset \ldots$ of weak hyper filters of H. We define the fuzzy weak hyper filter of μ of H by

$$\mu(x)=\begin{cases} 0 & \text{if} \quad x \notin F_n \quad \text{for each } n \in \mathbb{N}; \\ \dfrac{1}{n} & \text{if} \quad x \in F_n - F_{n-1} \quad \text{for } n = 1,2,\ldots; \end{cases}$$

where $F_0 = \emptyset$. By Theorem 3.6, μ is a fuzzy weak hyper filter of H, but $Im(\mu)$ is not a well-ordered subset of $[0,1]$. We get a contradiction. Consequently, H is Noetherian.

Corollary 3.17 Let H be a hyper BE-algebra. If for every fuzzy weak hyper filter μ of H such that $\mathrm{Im}(\mu)$ is a finite set, then H is Noetherian.

Theorem 3.18 Let H be a hyper BE-algebra and $T=\{t_1, t_2, \ldots\}\cup\{0\}$, where $\{t_n\}_{n=1}^{\infty}$ is a strictly decreasing in $[0,1]$. Then the following conditions are equivalent:

(i) H is Noetherian;

(ii) for every fuzzy weak hyper filter μ of H, if $\mathrm{Im}(\mu) \subseteq \mathrm{T}$, then there exists $k\in\mathbb{N}$ such that $Im(\mu) \subseteq \{t_1, t_2, \ldots, t_k\}\cup\{0\}$.

Proof. (i) $\Rightarrow$ (ii): Assume that H is a Noetherian. Let μ be a fuzzy weak hyper filter of H such that $\mathrm{Im}(\mu) \subseteq \mathrm{T}$. By Theorem 3.16, $\mathrm{Im}(\mu)$ is a well-ordered subset of $[0,1]$. Hence, there exists $k\in\mathbb{N}$ such that $Im(\mu) \subseteq \{t_1, t_2, \ldots, t_k\}\cup\{0\}$.

(ii) $\Rightarrow$ (i): Assume that for every fuzzy weak hyper filter μ of H, if $\mathrm{Im}(\mu) \subseteq \mathrm{T}$, then there exists $k\in\mathbb{N}$ such that $Im(\mu) \subseteq \{t_1, t_2, \ldots, t_k\}\cup\{0\}$. Suppose that H is not Noetherian. Then there exists a strictly ascending chain $F_1 \subset F_2 \subset F_3 \subset \ldots$ of weak hyper filters of H. We define a fuzzy set μ of H by

$$\mu(x)=\begin{cases} 0 & \text{if} \quad x \notin F_n \quad \text{for each } n \in \mathbb{N}; \\ t_n & \text{if} \quad x \in F_n - F_{n-1} \quad \text{for } n = 1,2,\ldots; \end{cases}$$

where $F_0 = \emptyset$. By Theorem 3.6, μ is a fuzzy weak hyper filter of H. This is a contradiction with our assumption. Therefore, H is Noetherian.

Theorem 3.19 Let H be a hyper BE-algebra and $T=\{t_1, t_2, \ldots\}\cup\{0\}$, where $\{t_n\}_{n=1}^{\infty}$ is a strictly increasing sequence in $[0,1]$. Then the following conditions are equivalent:

(i) H is Artinian;

(ii) for every fuzzy weak hyper filter μ of H, if $\mathrm{Im}(\mu) \subseteq \mathrm{T}$, then there exists $k\in\mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\}\cup\{0\}$.

Proof. (i) $\Rightarrow$ (ii): Assume that H is Artinian. Let μ be a fuzzy weak hyper filter of H such that $\mathrm{Im}(\mu) \subseteq \mathrm{T}$. Suppose that $t_{i_1} < t_{i_2} < \cdots < t_{i_m} < \ldots$ is a strictly increasing sequence of elements in $\mathrm{Im}(\mu)$. Let $I_m = U(\mu;\, t_{i_m})$ for $m=1,2,\ldots$ This implies that $I_1 \supset I_2 \supset \ldots \supset I_m \supset \ldots$ is a strictly descending chain of weak hyper filters μ of H, which is a contradiction that H is Artinian.

(ii) $\Rightarrow$ (i): Assume that for every fuzzy weak hyper filter μ of H, if $\mathrm{Im}(\mu) \subseteq \mathrm{T}$, then there exists $k\in\mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\}\cup\{0\}$. Suppose that H is not Artinian. Then there exists a strictly descending chain $F_1 \supset F_2 \supset \ldots \supset F_n \supset \ldots$ of weak hyper filters of H. We define a fuzzy set μ in H by

$$\mu(x)=\begin{cases} 0 & \text{if} \quad x \notin F_1, \\ t_n & \text{if} \quad x \in F_n - F_{n+1} \quad \text{for } n = 1,2,\ldots, \\ 1 & \text{if} \quad x \in F_n \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

We have that $\mu(1) = 1 \geq \mu(x)$, for all $x\in H$. Next, let $x,y\in H$. Thus, we can divide to be three cases, as follows.

Case 1: $x \notin F_1$. Then $y \circ x \notin F_1$ or $y \notin F_1$. Thus, Case 2: $x \in F_n - F_{n+1}$ for some $n=1,2,\ldots$. Then $y \circ x \nsubseteq F_{n+1}$ or $y \notin F_{n+1}$. We obtain that $\mu(y) \leq t_n$ or $\mu(z) \leq t_n$ for some $z \in y \circ x \backslash F_{n+1}$. So, $\min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\} \leq t_n = \mu(x)$..

Case 3: $x \in F_n$ for all $n \in \mathbb{N}$. Clearly, $\mu(x) = 1 \geq \min\{\inf_{z \in y \circ x} \mu(z), \mu(y)\}$.

Hence, μ is a fuzzy weak hyper filter of H. We have a contradiction with our assumption. Consequently, H is Artinian.

Corollary 3.20 Let H be a hyper BE-algebra. If for every fuzzy weak hyper filter μ of H, $\mathrm{Im}(\mu)$ is a finite set, then H is Artinian.

Conclusions

The concept of fuzzy weak hyper filters in hyper BE-algebras is introduced and investigated. It was shown that the set of all fuzzy weak hyper filters of hyper BE-algebras is a distributive complete lattice. Also, the concepts of Noetherian hyper BE-algebras and Artinian hyper BE-algebras are characterized by their fuzzy weak hyper filters. In future work, we will study the concept of characterizations of fuzzy weak hyper filters in hyper BE-algebras.

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