

สมบัติของตัวดำเนินการ R^a_m ในปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติ

Properties of R^a_m -operator in minimal structure space with an ideal

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บทคัดย่อ

ในบทความนี้ ผู้จัดได้นำเสนอเซตเปิดแบบ δ - m เชตเปิดแบบ a - m พังก์ชันแบบ δ - m -local ตัวดำเนินการแบบ R^a_m บนปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติพร้อมทั้งศึกษาสมบัติของพังก์ชัน และตัวดำเนินการนี้

คำสำคัญ: เชตเปิดแบบ δ - m เชตเปิดแบบ a - m พังก์ชันแบบ δ - m -local ตัวดำเนินการแบบ R^a_m ปริภูมิโครงสร้างเล็กสุดที่มีอุดมคติ

Abstract

In this article, the concepts of δ - m -open sets, a - m -open sets in a minimal structure space with an ideal are introduced. In addition, we present an a - m -local function and an R^a_m -operator in a minimal structure space with an ideal. We studied the properties of the function and this operator.

Keywords: δ - m -open sets, a - m -open sets, δ - m -local functions, R^a_m -operator, a minimal structure space with an ideal.

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Introduction

In 1945, Vaidyanathaswamy (1945) defined a local function in an ideal topological space and studied some properties of this function. In 1996, Maki, Umehara and Noiri (1996) defined a minimal structure and studied some properties of this structure. In 2014, Al-Omeri *et al.* (2014) defined an *a-local* function in an ideal topological space and also studied some properties of an *a-local* function. Later in 2016, Al-Omeri *et al.* (2016) defined an R_a -operator in an ideal topological space and studied some properties of this operator. In this article, we introduce the concepts of δ -*m-open* sets and δ -*m-open* sets in a minimal structure space with an ideal and study some fundamental properties. Moreover, we introduce the notions of δ -*m-local* functions and R_m^a -operators in minimal structure spaces, along with studying some properties related to an δ -*m-local* function and an R_m^a -operator defined above.

Preliminaries

Definition 2.1⁵ Let X be a nonempty set and $P(X)$ the power set of X . A subfamily m of $P(X)$ is called a minimal structure (briefly *MS*) on X if $\emptyset \in m$ and $X \in m$.

By (X, m) we denote a nonempty set X with a minimal structure m on X and it is called a minimal structure space. Each member of m is said to be *m-open* and the complement of *m-open* is said to be *m-closed*.

Definition 2.2 (Noiri & Popa, 2009) Let (X, m) be a minimal structure space and $A \subseteq X$. The *m-closure* of A , denoted by $CI_m(A)$ and the *m-interior* of A , denoted by $Int_m(A)$, are defined as follows ;

- 1) $CI_m(A) = \bigcap \{F : A \subseteq F, X \setminus F \in m\},$
- 2) $Int_m(A) = \bigcup \{U : U \subseteq A, U \in m\}.$

Lemma 2.3 (Maki & Gani, 1999) Let (X, m) be a minimal structure space and $A, B \subseteq X$, the following properties hold ;

(1) $CI_m(X \setminus A) = X \setminus Int_m(A)$ and $Int_m(X \setminus A) = X \setminus CI_m(A)$.

(2) If $X \setminus A \in m$, then $CI_m(A) = A$ and if $A \in m$, then $Int_m(A) = A$.

(3) $CI_m(\emptyset) = \emptyset$, $CI_m(X) = X$, $Int_m(\emptyset) = \emptyset$, and $Int_m(X) = X$.

(4) If $A \subseteq B$, then $CI_m(A) \subseteq CI_m(B)$ and $Int_m(A) \subseteq Int_m(B)$.

(5) $A \subseteq CI_m(A)$ and $Int_m(A) \subseteq A$.

(6) $CI_m(CI_m(A)) = CI_m(A)$ and $Int_m(Int_m(A)) = Int_m(A)$.

Lemma 2.4 (Maki & Gani, 1999) Let (X, m) be a minimal structure space and $A \subseteq X, x \in X$. Then $x \in CI_m(A)$ if and only if $U \cap A \neq \emptyset$ for every an *m-open* set U containing X .

Definition 2.5 (Rosas *et al.*, 2009) Let (X, m) be a minimal structure space and $A \subseteq X$.

(1) A is called *m-regular open* if $A = Int_m(CI_m(A))$

(2) A is called *m-regular closed* if $X \setminus A$ is *m-regular open*.

The family of all *m-regular open* sets of X is denoted by $r(m)$ and the family of all *m-regular closed* sets of X is denoted by $rc(m)$.

Definition 2.6 (Ozbakir & Yildirim, 2009) An ideal \mathcal{I} on a minimal structure space (X, m) is a nonempty collection of subsets of X which satisfies the following properties ;

- (1) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$ (heredity),
- (2) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (finite additivity).

The set \mathcal{I} together with a minimal structure space (X, m) is called a minimal structure space with an ideal, denoted by (X, m, \mathcal{I}) .

Main Results

Definition 3.1 Let (X, m) be a minimal structure space. A subset A is said to be *δ -m-open* if for each $X \in A$ there exists an *m-regular* open set G such that $X \in G \subseteq A$. The complement of *δ -m-open* set is called *δ -m-closed*. The family of all *δ -m-closed* sets of X , denoted by $\delta C_m(X)$.

Theorem 3.2 Let (X, m) be a minimal structure space and $A \subseteq X$. The arbitrary union of *δ -m-open* sets is a *δ -m-open* set.

Proof Let B_α be a *δ -m-open* set for all $\alpha \in J$ where J is an index set and let $x \in \bigcup_{\alpha \in J} B_\alpha$. There exists $\beta \in J$ such that $x \in B_\beta$. Since B_β is *δ -m-open*, there exists an *m-regular* open set G_β such that $X \in G_\beta \subseteq B_\beta$. Then $x \in G_\beta \subseteq B_\beta \subseteq \bigcup_{\alpha \in J} B_\alpha$. Therefore $\bigcup_{\alpha \in J} B_\alpha$ is *δ -m-open*.

Definition 3.3 Let (X, m) be a minimal structure space and $A \subseteq X$. A point $x \in X$ is called a δ -*m-cluster* point of A if $U \cap A \neq \emptyset$ for each m -regular open set U containing x .

Definition 3.4 Let (X, m) be a minimal structure space and $A \subseteq X$. The set of all δ -*m-cluster* points of A is called δ -*m-closure* of A and is denoted by $C_{\delta m}(A)$ and the union m -regular open sets contained in A is called the δ -*m-interior* of A , denoted by $I_{\delta m}(A)$.

Theorem 3.5 Let (X, m) be a minimal structure space and $A \subseteq X$. Then A is δ -*m-open* if and only if $I_{\delta m}(A) = A$.

Proof (\Rightarrow) Suppose that A is δ -*m-open*. By definition of δ -*m-interior*, $I_{\delta m}(A) = A$. Let $x \in A$. Since A is δ -*m-open*, there exists an m -regular open set O such that $x \in O \subseteq A$. This implies that $x \in I_{\delta m}(A)$. Then $A \subseteq I_{\delta m}(A)$. Hence $A = I_{\delta m}(A) = A$. (\Leftarrow) It follows from Theorem 3.2.

Theorem 3.6 Let (X, m) be a minimal structure space and $A, B \subseteq X$. The following property hold ;

- (1) If $A \subseteq B$, then $I_{\delta m}(A) \subseteq I_{\delta m}(B)$,
- (2) If $A \subseteq B$, then $C_{\delta m}(A) \subseteq C_{\delta m}(B)$.

Proof (1) Assume that $A \subseteq B$ and $x \in I_{\delta m}(A)$. Then, there exists an m -regular open set G such that $x \in G \subseteq A$. Since $A \subseteq B$, we have $x \in G \subseteq A \subseteq B$. This implies that $x \in I_{\delta m}(B)$. Hence $I_{\delta m}(A) \subseteq I_{\delta m}(B)$.

(2) Let $A \subseteq B$. Assume that $x \notin C_{\delta m}(B)$. Then there exists an m -regular open set U containing x such that $U \cap B = \emptyset$. Since $A \subseteq B$, we have $U \cap A \subseteq U \cap B = \emptyset$. Thus $x \notin C_{\delta m}(A)$. Therefore $C_{\delta m}(A) \subseteq C_{\delta m}(B)$.

Theorem 3.7 Let (X, m) be a minimal structure space and $A \subseteq X$. The following properties hold ;

- (1) $C_{\delta m}(A) = X \setminus I_{\delta m}(X \setminus A)$,
- (2) $I_{\delta m}(A) = X \setminus C_{\delta m}(X \setminus A)$.

Proof (1) We will show that $C_{\delta m}(A) = X \setminus I_{\delta m}(X \setminus A)$ by contrapositive. Assume that $x \notin X \setminus I_{\delta m}(X \setminus A)$. We get that $X \setminus I_{\delta m}(X \setminus A)$. So there exists an m -regular open set G such that $x \in G \subseteq X \setminus A$. Then $G \cap A = \emptyset$ and $x \notin C_{\delta m}(A)$. Thus $C_{\delta m}(A) \subseteq X \setminus I_{\delta m}(X \setminus A)$.

Next, we show that $X \setminus I_{\delta m}(X \setminus A) \subseteq C_{\delta m}(A)$ by contrapositive. Assume that $x \notin C_{\delta m}(A)$. Then x is not a δ -*m-cluster* point of A . There exists an m -regular open set G containing x such that $G \cap A = \emptyset$.

So $x \in G \subseteq X \setminus A$ and we get that $x \in I_{\delta m}(X \setminus A)$. Hence $x \notin X \setminus I_{\delta m}(X \setminus A)$. Thus $X \setminus I_{\delta m}(X \setminus A) \subseteq C_{\delta m}(A)$.

(2) Since $X \setminus A \subseteq X$, we have $C_{\delta m}(X \setminus A) = X \setminus I_{\delta m}(X \setminus (X \setminus A))$ by (1) and we get $C_{\delta m}(X \setminus A) = X \setminus I_{\delta m}(A)$. Therefore $I_{\delta m}(X \setminus A) = X \setminus C_{\delta m}(X \setminus A)$.

Definition 3.8 Let (X, m) be a minimal structure space and $A \subseteq X$.

(1) A is called *a-m-open* if $A \subseteq \text{Int}_m(CI_m(I_{\delta m}(A)))$. The family of all *a-m-open* sets of X is denoted by \mathcal{M}^a .

(2) A is called *a-m-closed* if $CI_{\delta m}(\text{Int}_m(C_{\delta m}(A))) \subseteq A$.

Theorem 3.9 Let (X, m) be a minimal structure space and $A \subseteq X$. Then A is *a-m-open* if and only if $X \setminus A$ is *a-m-closed*.

Proof Assume that A is *a-m-open*. Then $A \subseteq \text{Int}_m(CI_m(I_{\delta m}(A)))$, and $X \setminus A \supseteq X \setminus (\text{Int}_m(CI_m(I_{\delta m}(A))))$. By Lemma 2.3 and Theorem 3.7, $X \setminus A \supseteq CI_m(\text{Int}_m(C_{\delta m}(X \setminus A)))$. Therefore, $X \setminus A$ is *a-m-closed*.

Conversely, assume that $X \setminus A$ is *a-m-closed*.

Then $CI_m(\text{Int}_m(C_{\delta m}(X \setminus A))) \subseteq X \setminus A$ and $X \setminus CI_m(\text{Int}_m(C_{\delta m}(X \setminus A))) \supseteq X \setminus (X \setminus A)$. By Lemma 2.3 and Theorem 3.7, $\text{Int}_m(CI_m(I_{\delta m}(A))) \supseteq A$. Hence A is *a-m-open*.

Example 3.10 Let $X = \{a, b, c, d\}$ with a minimal structure $m = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, x\}$. Then $r(m) = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, x\}$, and $\delta O_m(x) = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, x\}$. $\mathcal{M}^a = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, x\}$. In this example $\{a, b\}, \{a, d\} \in \mathcal{M}^a$ but $\{a, b\} \cap \{a, d\} = \{a\} \notin \mathcal{M}^a$, that means \mathcal{M}^a does not have the property that any finite intersection of *a-m-open* sets is *a-m-open*.

Definition 3.11 Let (X, m) be a minimal structure space and $A \subseteq X$. The *a-m-closure* of A , denoted by $aC_m(A)$ and the *a-m-interior* of A , denoted by $aI_m(A)$, are defined as follows ;

- (1) $aC_m(A) = \cap \{F : X \setminus F \in \mathcal{M}^a \text{ and } A \subseteq F\}$,
- (2) $aI_m(A) = \cup \{U : U \in \mathcal{M}^a \text{ and } U \subseteq A\}$.

Theorem 3.12 Let (X, m) be a minimal structure space and $A \subseteq X$, $x \in X$, Then $x \in aC_m(A)$ if and only if $U \cap A \neq \emptyset$ for every *a-m-open* set U containing x .

Proof (\Rightarrow) Suppose that there exists an *a-m-open* set U containing x such that $U \cap A = \emptyset$. So $A \subseteq X \setminus U$ and $X \setminus U$ is *a-m-closed*. Since $aC_m(A)$ is the intersection of

all a -closed sets containing A , $aC_m(A) \subseteq X \setminus U$. Since $x \notin X \setminus U$, we have $x \notin aC_m(A)$.

(\Leftarrow) Assume that $x \notin aC_m(A)$. Then there exists an a -closed set F such that $A \subseteq F$ and $x \notin F$. Choose $U = X \setminus F$. Then U is a -open and $x \in X \setminus F = U$. Moreover, $U \cap A \subseteq (X \setminus F) \cap F = \emptyset$.

Theorem 3.13 Let (X, m) be a minimal structure space and $A, B \subseteq X$. The following properties hold :

(1) If $A \subseteq B$, then $aC_m(A) \subseteq aC_m(B)$.

(2) If $A \subseteq B$, then $aI_m(A) \subseteq aI_m(B)$.

Proof (1) Assume that $A \subseteq B$ and $x \notin aC_m(B)$. Then there exists an a -open set U containing x such that $U \cap A = \emptyset$. Since $A \subseteq B$, $U \cap B = \emptyset$. Hence $x \notin aC_m(A)$.

(2) Let $A \subseteq B$ and $x \in aI_m(A)$. Then there exists an a -open set U such that $x \in U \subseteq A$. Since $A \subseteq B$, $x \in U \subseteq B$. Therefore $x \in aI_m(B)$.

Proposition 3.14 Let (X, m) be a minimal structure space. Then $\emptyset \in \mathcal{M}^a$ and $X \in \mathcal{M}^a$.

Proof Since $\emptyset \subseteq \text{Int}_m(CI_m(I_{\delta m}(\emptyset)))$, \emptyset is a -open, and so $\emptyset \in \mathcal{M}^a$. Clearly $X = \text{Int}_m(CI_m(X))$, so X is an m -regular open. Then X is δ - m -open, that is $I_{\delta m}(X) = X$, and so $X \subseteq \text{Int}_m(CI_m(I_{\delta m}(X)))$. Therefore $X \in \mathcal{M}^a$.

Theorem 3.15 Let (X, m) be a minimal structure space. Then the arbitrary union of elements of \mathcal{M}^a belongs to \mathcal{M}^a .

Proof Let V_α be a -open for all $\alpha \in J$ and $G = \bigcup_{\alpha \in J} V_\alpha$. Then $V_\alpha \subseteq \text{Int}_m(CI_m(I_{\delta m}(V_\alpha)))$ for all $\alpha \in J$. Since $V_\alpha \subseteq G$, it follows that $I_{\delta m}(V_\alpha) \subseteq I_{\delta m}(G)$ and so $CI_m(I_{\delta m}(V_\alpha)) \subseteq CI_m(I_{\delta m}(G))$. Then $\text{Int}_m(CI_m(I_{\delta m}(V_\alpha))) \subseteq \text{Int}_m(CI_m(I_{\delta m}(G)))$. This implies that $V_\alpha \subseteq \text{Int}_m(CI_m(I_{\delta m}(G)))$ for all $\alpha \in J$. Thus $\bigcup_{\alpha \in J} V_\alpha \subseteq \text{Int}_m(CI_m(I_{\delta m}(G)))$. Therefore $G \subseteq \text{Int}_m(CI_m(I_{\delta m}(G)))$.

Corollary 3.16 Let (X, m) be a minimal structure space. Then the arbitrary intersection of a -closed sets is an a -closed set.

Proof Let G_α be a -closed for all $\alpha \in J$. Then $X \setminus G_\alpha$ is a -open and so $\bigcup_{\alpha \in J} (X \setminus G_\alpha)$ is a -open. Since $X \setminus \bigcap_{\alpha \in J} G_\alpha = \bigcup_{\alpha \in J} (X \setminus G_\alpha)$, $\bigcap_{\alpha \in J} G_\alpha$ is a -closed.

Remark 3.17 In a minimal structure space, by Corollary 3.16, $aC_m(A)$ is a -closed.

Theorem 3.18 Let (X, m) be a minimal structure space and $A \subseteq X$. The following properties hold :

(1) $aC_m(aC_m(A)) = aC_m(A)$,

(2) $aI_m(aI_m(A)) = aI_m(A)$.

Proof (1) Clearly $aC_m(A) \subseteq aC_m(aC_m(A))$. Since $aC_m(A)$ is a -closed, $aC_m(aC_m(A)) \subseteq aC_m(A)$. Therefore $aC_m(aC_m(A)) = aC_m(A)$.

(2) Clearly $aI_m(aI_m(A)) = aI_m(A)$. Since $aI_m(A)$ is a -open, $aI_m(A) \subseteq aI_m(aI_m(A))$. Therefore $aI_m(aI_m(A)) = aI_m(A)$.

Let (X, m, \mathcal{I}) be a minimal structure space with an ideal. For each $x \in X$, let $\mathcal{M}^a(x) = \{U : x \in U, U \in \mathcal{M}^a\}$ be the family of all a -open sets that contain x .

Definition 3.19 Let (X, m, \mathcal{I}) be a minimal structure space with an ideal and $A \subseteq X$. Then $A_m^a(\mathcal{I}, m) = \{x \in X : U \cap A \notin \mathcal{I}, \text{ for every } U \in \mathcal{M}^a(x)\}$ is called a - m -local function of A with respect to \mathcal{I} and m . We denote simply A_m^a for $A_m^a(\mathcal{I}, m)$.

Remark 3.20 The minimal ideal is $\{\emptyset\}$ and the maximal ideal is $P(x)$ in any minimal structure space with an ideal (X, m, \mathcal{I}) . It can be deduced that $A_m^a(\{\emptyset\}, m) = aC_m(A)$ and $A_m^a(P(X), m) = \emptyset$ for every $A \subseteq X$.

Remark 3.21 In general, $A \not\subseteq A_m^a$ and $A_m^a \not\subseteq A$. The next example shows that $A \not\subseteq A_m^a$.

Example 3.22 Let $X = \{a, b, c, d\}$ with a minimal structure $m = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a, b\}$. Then $\mathcal{M}^a = \{\emptyset, \{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ and $A_m^a = \emptyset$.

Theorem 3.23 Let (X, m, \mathcal{I}) be a minimal structure space with an ideal and $A, B \subseteq X$. The following properties hold :

(1) $(\emptyset)_m^a = \emptyset$.

(2) If $A \subseteq B$, then $A_m^a \subseteq B_m^a$.

(3) $(A_m^a)_m^a \subseteq A_m^a$.

(4) $A_m^a \cup B_m^a \subseteq (A \cup B)_m^a$.

(5) $(A \cap B)_m^a \subseteq A_m^a \cap B_m^a$.

(6) $(A \setminus B)_m^a \setminus (B)_m^a \subseteq A_m^a \setminus B_m^a$.

Proof (1) Assume $(\emptyset)_m^a \neq \emptyset$. Then there exists $x \in (\emptyset)_m^a$. Since $X \in \mathcal{M}^a(X)$, $X \cap \emptyset \notin \mathcal{I}$. It contradicts with $X \cap \emptyset = \emptyset \notin \mathcal{I}$. Therefore $(\emptyset)_m^a = \emptyset$.

(2) Assume that $A \subseteq B$. We will show that $A_m^a \subseteq B_m^a$ by contrapositive. Suppose that $x \notin B_m^a$. Then there exists $U \in \mathcal{M}^a(X)$ such that $U \cap B \in \mathcal{I}$. From $A \subseteq B$ and the property of \mathcal{I} , $U \cap A \in \mathcal{I}$. Therefore $x \notin A_m^a$.

(3) Assume that $x \in (A_m^a)_m^a$, and $U \in \mathcal{M}^a(X)$. Then $A_m^a \cap U \notin \mathcal{I}$ and so $A_m^a \cap U \neq \emptyset$. Thus there exists $y \in A_m^a \cap U$, and so $y \in U \in \mathcal{M}^a(y)$. This implies that $A \cap U \notin \mathcal{I}$. Therefore $x \in A_m^a$.

(4) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (2) $A_m^a \subseteq (A \cup B)_m^a$ and $B_m^a \subseteq (A \cup B)_m^a$. So $A_m^a \cup B_m^a \subseteq (A \cup B)_m^a$.

(5) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, by (2) $(A \cap B)_m^a \subseteq A_m^a$ and $(A \cap B)_m^a \subseteq B_m^a$. So $(A \cap B)_m^a \subseteq A_m^a \cap B_m^a$.

(6) Since $A \setminus B \subseteq A$, by (2) $(A \setminus B)_m^a \subseteq A_m^a$. So $(A \setminus B)_m^a \setminus B_m^a \subseteq A_m^a \setminus B_m^a$.

Theorem 3.24 Let (X, m) be a minimal structure space and \mathcal{I}, \mathcal{J} are ideals on X where $\mathcal{I} \subseteq \mathcal{J}$. Then $A_m^a(\mathcal{J}, m) \subseteq A_m^a(\mathcal{I}, m)$ for all $A \subseteq X$.

Proof Let $A \subseteq X$. Assume that $x \in A_m^a(\mathcal{J}, m)$. Then $U \cap A \notin \mathcal{J}$ for every $U \in \mathcal{M}^a(x)$. Since $\mathcal{I} \subseteq \mathcal{J}$, $U \cap A \notin \mathcal{I}$ for every $U \in \mathcal{M}^a(x)$. Thus $x \in A_m^a(\mathcal{I}, m)$. Hence $A_m^a(\mathcal{J}, m) \subseteq A_m^a(\mathcal{I}, m)$.

Theorem 3.25 Let (X, m, \mathcal{I}) be a minimal structure space with an ideal and $A \subseteq X$. The following properties hold :

$$(1) A_m^a \subseteq aC_m(A),$$

(2) $A_m^a = aC_m(A)$, (i.e., A_m^a is an a -m-closed subset).

Proof (1) Assume that $x \notin aC_m(A)$. Then there exists an a -m-closed set F such that $A \subseteq F$ and $x \notin F$. Thus $x \in X \setminus F$, and so $X \setminus F \in \mathcal{M}^a(x)$. Hence $(X \setminus F) \cap A = \emptyset \in \mathcal{I}$, and so $x \notin A_m^a$. This implies that $A_m^a \subseteq aC_m(A)$.

(2) It is clear that $A_m^a \subseteq aC_m(A_m^a)$. Next, we will prove that $aC_m(A_m^a) \subseteq A_m^a$. Let $x \in aC_m(A_m^a)$ and $U \in \mathcal{M}^a(x)$. Then $A_m^a \cap U \neq \emptyset$. Therefore there exists $y \in A_m^a \cap U$, so $U \in \mathcal{M}^a(y)$. Since $y \in A_m^a$, $A \cap U \notin \mathcal{I}$, and so $x \in A_m^a$. Then $A_m^a = aC_m(A_m^a)$.

Theorem 3.26 Let (X, m, \mathcal{I}) be a minimal structure space with an ideal and $A \subseteq X$. The following properties hold :

$$(1) \text{ If } A \in \mathcal{I}, \text{ then } A_m^a = \emptyset.$$

$$(2) \text{ If } U \in \mathcal{I}, \text{ then } A_m^a = (A \cup U)_m^a.$$

$$(3) \text{ If } U \in \mathcal{I}, \text{ then } A_m^a = (A \setminus U)_m^a.$$

Proof (1) Assume that $A_m^a \neq \emptyset$. Then there exists $x \in A_m^a$. Since $X \in \mathcal{M}^a(x)$, $A \cap X \notin \mathcal{I}$.

(2) Assume that $U \in \mathcal{I}$. Since $A \subseteq A \cup U$ by Theorem 3.23(2), we get $A_m^a \subseteq (A \cup U)_m^a$. Next, we will prove that $(A \cup U)_m^a \subseteq A_m^a$ by contrapositive. Suppose that $x \notin A_m^a$. Then there exists $V \in \mathcal{M}^a(x)$ such that $A \cap V \in \mathcal{I}$. Since $(A \cup U) \cap V = (A \cap V) \cup (U \cap V) \in \mathcal{I}$, $(A \cup U) \cap V \in \mathcal{I}$. Therefore $x \notin (A \cup U)_m^a$.

(3) Assume that $U \in \mathcal{I}$. Since $A_m^a = (A \cap X)_m^a = (A \cap ((X \setminus U) \cup U))_m^a = ((A \setminus U) \cup (A \cup U))_m^a$ and $A \cap U \subseteq U \in \mathcal{I}$, by (2) $A_m^a = (A \setminus U)_m^a$.

Definition 3.27 Let (X, m, \mathcal{I}) be a minimal structure with an ideal. An operator $\mathfrak{R}_m^a : P(X) \rightarrow P(X)$ is defined as follows ; for every $A \in P(X)$, $\mathfrak{R}_m^a(A) = \{x \in X : \text{there exists } U \in \mathcal{M}^a(x) \text{ such that } U \cap A \in \mathcal{I}\}$.

Theorem 3.28 Let (X, m, \mathcal{I}) be a minimal structure space with an ideal and $A \in P(X)$. Then $\mathfrak{R}_m^a(A) = X \setminus (X \setminus A)_m^a$.

Proof Let $x \in \mathfrak{R}_m^a(A)$. Then there exists an a -m-open set U containing x such that $U \cap A \in \mathcal{I}$. Thus $U \cap (X \setminus A) \in \mathcal{I}$. So $x \notin (X \setminus A)_m^a$ and hence $x \in X \setminus (X \setminus A)_m^a$. Therefore $\mathfrak{R}_m^a(A) \subseteq X \setminus (X \setminus A)_m^a$.

For the reverse inclusion, let $x \in X \setminus (X \setminus A)_m^a$. Then $x \notin (X \setminus A)_m^a$. Thus there exists an a -m-open set U containing x such that $U \cap (X \setminus A) \in \mathcal{I}$. This implies that $U \cap A \in \mathcal{I}$. Hence $x \in \mathfrak{R}_m^a(A)$. So $X \setminus (X \setminus A)_m^a \subseteq \mathfrak{R}_m^a(A)$. Therefore $\mathfrak{R}_m^a(A) = X \setminus (X \setminus A)_m^a$.

Example 3.29 Let $X = \{a, b, c, d\}$ with a minimal structure $m = \{\emptyset, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $A = \{a, b\}$. Then $\mathfrak{R}_m^a(A) = \{a, b\}$.

Theorem 3.30 Let (X, m, \mathcal{I}) be a minimal structure space with an ideal, and $A \subseteq X$. Then $\mathfrak{R}_m^a(A)$ is a -m-open.

Proof We know that $\mathfrak{R}_m^a(A) = X \setminus (X \setminus A)_m^{a^*}$ and $(X \setminus A)_m^{a^*}$ is a -*m-closed*. Therefore $\mathfrak{R}_m^a(A)$ is a -*m-open*.

Theorem 3.31 Let (X, m, \mathcal{J}) be a minimal structure space with an ideal and $A, B \subseteq X$. Then the following properties hold ;

- (1) If $A \subseteq B$, then $\mathfrak{R}_m^a(A) \subseteq \mathfrak{R}_m^a(B)$.
- (2) If $A \subseteq B$, then $\mathfrak{R}_m^a(A \cap B) \subseteq \mathfrak{R}_m^a(A) \cap \mathfrak{R}_m^a(B)$.
- (3) If $A \subseteq B$, then $\mathfrak{R}_m^a(A) \cup \mathfrak{R}_m^a(B) \subseteq \mathfrak{R}_m^a(A \cup B)$.
- (4) If $A \in \mathcal{M}^a$, then $A \subseteq \mathfrak{R}_m^a(A)$.
- (5) If $A \subseteq B$, then $\mathfrak{R}_m^a(A) \subseteq \mathfrak{R}_m^a(\mathfrak{R}_m^a(A))$.

Proof (1) Assume that $A \subseteq B$. Then $X \setminus B \subseteq X \setminus A$. By Theorem 3.23(2), $(X \setminus B)_m^{a^*} \subseteq (X \setminus A)_m^{a^*}$ and hence $X \setminus (X \setminus A)_m^{a^*} \subseteq X \setminus (X \setminus B)_m^{a^*}$. Therefore $\mathfrak{R}_m^a(A) \subseteq \mathfrak{R}_m^a(B)$.

(2) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $\mathfrak{R}_m^a(A \cap B) \subseteq \mathfrak{R}_m^a(A)$ and $\mathfrak{R}_m^a(A \cap B) \subseteq \mathfrak{R}_m^a(B)$. Therefore $\mathfrak{R}_m^a(A \cap B) \subseteq \mathfrak{R}_m^a(A) \cap \mathfrak{R}_m^a(B)$.

(3) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $\mathfrak{R}_m^a(A) \subseteq \mathfrak{R}_m^a(A \cup B)$ and $\mathfrak{R}_m^a(B) \subseteq \mathfrak{R}_m^a(A \cup B)$. Therefore $\mathfrak{R}_m^a(A) \cup \mathfrak{R}_m^a(B) \subseteq \mathfrak{R}_m^a(A \cup B)$.

(4) Assume that $A \in \mathcal{M}^a$. Then $X \setminus A$ is a -*m-closed*. By Theorem 3.25(1), we get that $(X \setminus A)_m^{a^*} \subseteq aC_m(X \setminus A) = X \setminus A$. Therefore $A = X \setminus (X \setminus A) \subseteq X \setminus (X \setminus A)_m^{a^*} = \mathfrak{R}_m^a(A)$.

(5) By Theorem 3.30, we get that $\mathfrak{R}_m^a(A)$ is a -*m-open*. By (4), we get that $\mathfrak{R}_m^a(A) \subseteq \mathfrak{R}_m^a(\mathfrak{R}_m^a(A))$.

Theorem 3.32 Let (X, m, \mathcal{J}) be a minimal structure space with an ideal and $A, B, U \subseteq X$. Then the following properties hold ;

- (1) If $U \in \mathcal{J}$, then $\mathfrak{R}_m^a(A \setminus U) = \mathfrak{R}_m^a(A)$.
- (2) If $U \in \mathcal{J}$, then $\mathfrak{R}_m^a(A \cup U) = \mathfrak{R}_m^a(A)$.
- (3) If $(A \setminus B) \cup (B \setminus A) \in \mathcal{J}$, then $\mathfrak{R}_m^a(A) = \mathfrak{R}_m^a(B)$.
- (4) If $A \in \mathcal{J}$, then $\mathfrak{R}_m^a(A) = X \setminus X_m^{a^*}$.

Proof (1) Assume that $A \subseteq X$, $U \in \mathcal{J}$. By Theorem 3.26(2) and 3.28, we have $\mathfrak{R}_m^a(A \setminus U) = X \setminus (X \setminus (A \setminus U))_m^{a^*} = X \setminus ((X \setminus A) \cup U)_m^{a^*} = X \setminus (X \setminus A)_m^{a^*}$. Therefore $\mathfrak{R}_m^a(A \setminus U) = \mathfrak{R}_m^a(A)$.

(2) Assume that $U \in \mathcal{J}$. By Theorem 3.26(3), we have $\mathfrak{R}_m^a(A \cup U) = X \setminus (X \setminus (A \cup U))_m^{a^*} = X \setminus ((X \setminus A) \cup U)_m^{a^*} = X \setminus (X \setminus A)_m^{a^*} = \mathfrak{R}_m^a(A)$.

(3) Assume that $(A \setminus B) \cup (B \setminus A) \in \mathcal{J}$.

Thus

$$\begin{aligned} \mathfrak{R}_m^a(A) &= \mathfrak{R}_m^a(A \setminus (A \setminus B)) \\ &= \mathfrak{R}_m^a((A \setminus (A \setminus B)) \cup (B \setminus A)) \\ &= \mathfrak{R}_m^a(B). \end{aligned}$$

(4) Assume that $A \in \mathcal{J}$. By Theorem 3.26(3), we get that $\mathfrak{R}_m^a(A) = X \setminus (X \setminus A)_m^{a^*} = X \setminus X_m^{a^*}$.

Theorem 3.33 Let (X, m, \mathcal{J}) be a minimal structure space with an ideal and $A \subseteq X$. Then $\mathfrak{R}_m^a(A) = \mathfrak{R}_m^a(\mathfrak{R}_m^a(A))$ if and only if $(X \setminus A)_m^{a^*} = ((X \setminus A)_m^{a^*})_m^{a^*}$.

Proof It follows from the facts that,

$$\begin{aligned} \text{i) } \mathfrak{R}_m^a(A) &= X \setminus (X \setminus A)_m^{a^*} \text{ and } \mathfrak{R}_m^a(\mathfrak{R}_m^a(A)) \\ \text{ii) } \mathfrak{R}_m^a(\mathfrak{R}_m^a(A)) &= X \setminus [X \setminus (X \setminus (X \setminus A)_m^{a^*})]_m^{a^*} \\ &= X \setminus ((X \setminus A)_m^{a^*})_m^{a^*}. \end{aligned}$$

Therefore $\mathfrak{R}_m^a(A) = \mathfrak{R}_m^a(\mathfrak{R}_m^a(A))$ if and only if $(X \setminus A)_m^{a^*} = ((X \setminus A)_m^{a^*})_m^{a^*}$.

Discussion and Conclusion

The aim of this article is to introduce the results of properties of some sets in a minimal structure space with an ideal. In addition, we study some properties of δ -*m-open* sets, *a*-*m-open* sets in a minimal structure space with an ideal. Moreover, we define an δ -*m-local* function and an R_m^a -operator in a minimal structure space with an ideal. Some properties of them are obtained.

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