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Mathematical Model of Transient Electromagnetic Response from Heterogeneous Ground Structure

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Abstract

In this paper, we study the earth's surface layers using time-domain electromagnetic field by constructing 2 mathematical models. In the first model, the ground is considered uniformly and having constant conductivity profile, denoted by a positive constant σ_0. The second model, we divide the ground surface into 2 laterally uniform layers. The conductivity of overburden increases with depth, given by $\sigma_0 e^{-b(z-d)}$, $0 \leq z \leq d$, where b is a positive constant, and d is the thickness of overburden. The conductivity of host medium, $z \geq d$, is a constant σ_0. By solving the models using mathematical techniques, the electric fields response from the ground surface are simulated and plotted to show their behaviors which decay rapidly depending on the conductivity of ground structure.

***Key Words*:** Time domain; Electromagnetic; Transient; Electric

Introduction

The simplest models for the preparation of curves for the transient electromagnetic field response from the earth's surface layers are considered. The earth structure usually can be denoted by horizontally stratified earth (Banerjee et al. 1980), where each layer having homogeneous and isotropic electrical properties. A separate analytical study is necessitated when the conductivity of a layer varies in a particular direction, say, with depth. Such situations occur in permeability and salinity tests of the overburden and ground water recharging schemes. A geological situation where one of the layers has a conductivity varying with depth locates in much of the area near the sea shore in the southern and eastern parts of Thailand. In this paper, we have to define the nature of this study in order to use the time-domain electromagnetic response in heterogeneous media. The conductivity of the ground depends on the depth only. Firstly, we consider the ground which has a constant conductivity profile, defined by $\sigma_p(z) = \sigma_0$, where σ_0 is a positive constant. Secondly, we consider the ground containing two layers, the overburden and the host medium. The conductivity of overburden given by $\sigma_{ove}(z) = \sigma_0 e^{-b(z-d)}$, $0 \leq z \leq d$, where b is a positive constant and d is the thickness of overburden. The conductivity of host medium, $z \geq d$, is constant given by $\sigma_{host}(z) = \sigma_0$. The exponential ground profile used in this paper is different from the model used by Banerjee et al. (1980), Kim and Lee (1996), Lee and Ignetik (1994), Yooyuanyong and Siew (2000), and Yooyuanyong (1999, 2000). The objective of this paper is to present mathematical models and techniques for studying the structures of the earth's surface layers. The electric fields from the two models are then compared and discussed.

Formulation of the Problems

In cylindrical coordinates with the basis vectors, $\bar{e}_r$, $\bar{e}_\theta$ and $\bar{e}_z$, let r, θ, and z denote the axes along the basis vectors, respectively. For the horizontal loop source, the azimuthally symmetry gives electric fields $\bar{E}$ which are independent of the azimuth angle θ and it can be shown that the electric fields have only the E_θ component (Morrison et al., 1969; Ignetik et al., 1985), which we shall denote simply by $\bar{E}$as follows,

$$\bar{E} = E\bar{e}_\theta .$$

The magnetic fields have only the radial H_r and vertical H_z components. Following Morrison et al. (1969), it is accepted here to use exp ($-i\omega t$) dependence. Hence, these field quantities are found to satisfy the Maxwell's equations (Hohmann and Raiche, 1988) in the forms of

$$i\omega\mu H_r = -\frac{\partial E}{\partial z}, \quad (1)$$

$$i\omega\mu H_z = \frac{1}{r}\frac{\partial (rE)}{\partial r}, \quad (2)$$

and $$\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} = (\sigma(z) - i\omega\varepsilon)E + J_s, \quad (3)$$

where $J_s = aI(\omega)\delta\,(r\text{-}a)\delta\,(z\text{+}h)/r$ is the source current density, ω is the angular frequency, δ is the delta function (Jeffrey, 1995) in the form of $\delta(x) = \begin{cases} \infty; x = 0 \\ 0; x \neq 0 \end{cases}$ for all real number x, $\sigma(z)$ is the electrical conductivity of medium depending on the depth only, ε is the electric permittivity of medium, and μ is the magnetic permeability of medium. Equations (1) - (3) can be solved to get the differential equation for electric fields as

$$i\omega\mu J_s = -\frac{\partial^2 E}{\partial z^2} - \frac{\partial^2 E}{\partial r^2} - \frac{1}{r}\frac{\partial E}{\partial r} + \frac{E}{r^2} - (i\omega\mu\sigma(z) + \omega^2\mu\varepsilon)E. \quad (4)$$

Taking Hankel transform which is defined as

$$\tilde{E}(\lambda, z, \omega) = \int_0^\infty rJ_1(\lambda r)E(r, z, \omega)dr,$$

where J_1 is Bessel function of the first kind of order 1, and thus, equation (4) becomes

$$i\omega\mu a I(\omega)\delta(z+h)J_1(\lambda a) = -\frac{\partial^2 \tilde{E}}{\partial z^2} + (\lambda^2 - (i\omega\mu\sigma(z) + \omega^2\mu\varepsilon)\tilde{E}. \quad (5)$$

Half-Space Model with a Constant Conductivity Ground Profile

We now consider the ground with a constant conductivity profile which can be defined by $\sigma_p(z) = \sigma_0$, $z>0$, where σ_0 is a positive constant. This model represents to the rock area located near the mountain. There is a primary alternating source current carried by a coil of radius a, at $z = -h$ above the surface of the earth, $z = 0$ (see Figure 1).

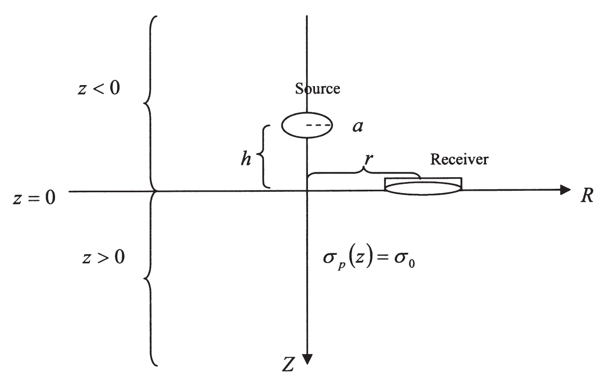

Figure 1 Illustration of the half-space model with a constant conductivity ground profile

The Electric Fields in Air

The electric fields in air can be denoted by $\tilde{E}_{air}(\lambda, z, \omega)$ and expressed as the sum of two components,

$$\tilde{E}_{air}(\lambda, z, \omega) = \tilde{E}^p(\lambda, z, \omega) + \tilde{E}^s(\lambda, z, \omega),$$

where $\tilde{E}^p(\lambda, z, \omega)$ is the primary field and $\tilde{E}^s(\lambda, z, \omega)$ is the secondary field. In air, $\sigma_{air}(z) \cong 0$ the magnetic permeability is approximated to be μ_0, where μ_0 is the magnetic permeability in free space and the electric permittivity is approximated to be ε_0, where ε_0 is the electric permittivity in free space. Therefore, the electric fields can be obtained from equation (5) as

$$i\omega\mu_0 aI(\omega)\delta(z+h)J_1(\lambda a)$$
$$= -\frac{\partial^2}{\partial z^2}\widetilde{E}^p(\lambda,z,\omega) + m_0^2\widetilde{E}^p(\lambda,z,\omega), \quad (6)$$

where $m_0^2 = \lambda^2 - k_0^2$ and $k_0^2 = \omega^2\mu_0\varepsilon_0$. Equation (6) is an ordinary differential equation and we can rewrite as

$$i\omega\mu_0 aI(\omega)\delta(z+h)J_1(\lambda a)$$
$$= -\frac{d^2}{dz^2}\widetilde{E}^p(\lambda,z,\omega) + m_0^2\widetilde{E}^p(\lambda,z,\omega). \quad (7)$$

Let $\widetilde{E}^p(\lambda,z,\omega) = \widetilde{E}_g^p(\lambda,z,\omega) + \widetilde{E}_h^p(\lambda,z,\omega),$ (8)

where $\widetilde{E}_g^p(\lambda,z,\omega)$ is the general solution of linear homogeneous ordinary differential equation and $\widetilde{E}_h^p(\lambda,z,\omega)$ is the particular solution. The general solution to equation (7) can be obtained from

$$\frac{d^2}{dz^2}\widetilde{E}_g^p(\lambda,z,\omega) - m_0^2\widetilde{E}_g^p(\lambda,z,\omega) = 0,$$

and the solution is

$$\widetilde{E}_g^p(\lambda,z,\omega) = A_1 e^{-m_0 z} + A_2 e^{m_0 z},$$

where both A_1 and A_2 are arbitrary constants to be determined from the boundary conditions. The particular solution of equation (7) can be achieved by using the method of variation of parameters to get the electric fields, $\widetilde{E}_h^p$, as

$$\widetilde{E}_h^p = -\frac{g}{2m_0}\int_{-\infty}^{0} G(z,x)dx,$$

where $g = -i\omega\mu_0 aI(\omega)\, J_1(\lambda a)$ and $G(z,x)$ is the Green's function defined by

$$G(z,x) = e^{-m_0|z-x|}\delta(x+h) = \begin{cases} e^{-m_0(z-x)}\delta(x+h); z \ge x \\ e^{m_0(z-x)}\delta(x+h); z < x \end{cases}.$$

Since $\int_{-\infty}^{0} e^{-m_0|z-x|}\delta(x+h)dx = e^{-m_0|z+h|}$, the electric fields, $\widetilde{E}_h^p$, can now be shown to be

$$\widetilde{E}_h^p(\lambda,z,\omega) = -\frac{g}{2m_0}e^{-m_0|z+h|}, \quad z \le 0.$$

The solution of equation (8) which represents the electric fields in air can be calculated from

$$\widetilde{E}^p(\lambda,z,\omega) = A_1 e^{-m_0 z} + A_2 e^{m_0 z}$$
$$+\frac{i\omega\mu_0 aI(\omega)J_1(\lambda a)e^{-m_0|z+h|}}{2m_0}, \quad z \le 0.$$

When $z \to -\infty$, $\widetilde{E}^p \to 0$, therefore, we require $A_1 = 0$. Hence,

$$\widetilde{E}^p(\lambda,z,\omega) = A_2 e^{m_0 z}$$
$$+\frac{i\omega\mu_0 aI(\omega)J_1(\lambda a)e^{-m_0|z+h|}}{2m_0}, \quad z \le 0 \quad (9)$$

For the secondary field, since $I(\omega) = 0$, equation (4) becomes

$$\frac{\partial^2}{\partial z^2}E^s(\lambda,z,\omega) + \frac{\partial^2}{\partial r^2}E^s(\lambda,z,\omega)$$
$$+\frac{1}{r}\frac{\partial}{\partial r}E^s(\lambda,z,\omega) - \frac{E^s(\lambda,z,\omega)}{r^2}$$
$$+k_0^2 E^s(\lambda,z,\omega) = 0. \quad (10)$$

Taking Hankel transform to equation (10), we obtain

$$\frac{\partial^2}{\partial z^2}\widetilde{E}^s(\lambda,z,\omega) - m_0^2\widetilde{E}^s(\lambda,z,\omega) = 0.$$

Since the above equation is a linear homogeneous ordinary differential equation, it can also be written as

$$\frac{d^2}{dz^2}\widetilde{E}^s(\lambda,z,\omega) - m_0^2\widetilde{E}^s(\lambda,z,\omega) = 0. \quad (11)$$

Furthermore, the general solution of equation (11) is

$$\widetilde{E}^s(\lambda,z,\omega) = A_3 e^{-m_0 z} + A_4 e^{m_0 z}, \quad (12)$$

where A_3 and A_4 are arbitrary constants to be determined from the boundary conditions. As $z \to -\infty$, then $\lim_{z\to-\infty}\widetilde{E}^s \to 0$, we require $A_3 = 0$ and hence the solution of equation (12) becomes

$$\widetilde{E}^s(\lambda,z,\omega) = A_4 e^{m_0 z}. \quad (13)$$

The electric fields in air now become

$$\widetilde{E}_{air}(\lambda,z,\omega) =$$
$$\frac{i\omega\mu_0 aI(\omega)J_1(\lambda a)e^{-m_0|z+h|}}{2m_0} + Ae^{m_0 z}, \quad z \le 0, \quad (14)$$

where $A = A_2 + A_4$.

The Electric Fields in Ground

From equation (4), we obtain the following partial differential equation

$$\frac{\partial^2}{\partial z^2} E_{gro}(r,z,\omega) + \frac{\partial^2}{\partial r^2} E_{gro}(r,z,\omega) + \frac{1}{r}\frac{\partial}{\partial r} E_{gro}(r,z,\omega) - \frac{E_{gro}(r,z,\omega)}{r^2} + k_g^2 E_{gro}(r,z,\omega) = 0, \quad (15)$$

where $E_{gro}(r,z,\omega)$ is the electric fields in ground, $k_g^2 = i\omega\mu_g\sigma_0 + \omega^2\mu_g\varepsilon_g$, μ_g is the magnetic permeability of ground, ε_g is the electric permittivity of ground, and σ_0 is the conductivity of ground which is assumed to be a constant. Taking Hankel transform to equation (15), we get

$$\frac{\partial^2}{\partial z^2} \tilde{E}_{gro}(\lambda,z,\omega) - m_g^2 \tilde{E}_{gro}(\lambda,z,\omega) = 0,$$

where $m_g^2 = \lambda^2 - k_g^2$ The above equation is a linear ordinary differential equation and we can write it as

$$\frac{d^2}{dz^2} \tilde{E}_{gro}(\lambda,z,\omega) - m_g^2 \tilde{E}_{gro}(\lambda,z,\omega) = 0, \quad (16)$$

The solution of equation (16) is

$$\tilde{E}_{gro}(\lambda,z,\omega) = A_5 e^{-m_g z} + A_6 e^{m_g z}, \quad (17)$$

where A_5 and A_6 are arbitrary constants to be determined from the boundary conditions. Under the condition that as $z \to \infty$, $\tilde{E}_{gro} \to 0$, we require $A_6 = 0$. Thus, equation (17) becomes

$$\tilde{E}_{gro}(\lambda,z,\omega) = A_5 e^{-m_g z}, \quad z \geq 0. \quad (18)$$

From now on, we assume that the magnetic permeability of the ground is μ_0, and the electric permittivity of the ground is ε_0. The arbitrary constants A and A_5 can be solved by imposing the continuity of $\tilde{E}$ and $\frac{\partial \tilde{E}}{\partial z}$ at air-earth interface. That is

$$\tilde{E}_{air}(\lambda,0,\omega) = \tilde{E}_{gro}(\lambda,0,\omega), \text{ and}$$

$$\frac{\partial}{\partial z}\tilde{E}_{air}(\lambda,0,\omega) = \frac{\partial}{\partial z}\tilde{E}_{gro}(\lambda,0,\omega).$$

We obtain the electric fields in air as

$$E_{air}(r,z,\omega) = \frac{i\omega\mu_0 a I(\omega)}{2}\int_0^\infty \frac{\lambda}{m_0}[e^{-m_0|z+h|} - e^{m_0(z-|h|)}]J_1(\lambda a)J_1(\lambda r)d\lambda + i\omega\mu_0 a I(\omega)\times \int_0^\infty [\frac{\lambda}{m_g+m_0}]J_1(\lambda a)J_1(\lambda r)e^{m_0(z-|h|)}d\lambda, \quad (19)$$

and the electric fields in ground are

$$E_{gro}(r,z,\omega) = \int_0^\infty \lambda\tilde{E}_{gro}(\lambda,z,\omega)J_1(\lambda r)d\lambda,$$

$$= i\omega\mu_0 a I(\omega)\int_0^\infty \frac{\lambda}{m_g+m_0}J_1(\lambda a)\times J_1(\lambda r)e^{-(m_g z + m_0|h|)}d\lambda. \quad (20)$$

The electric fields on the ground surface E_{sur}, can be determined from equation (19) or equation (20) by considering $z = 0$. Thus, we have

$$E_{sur}(r,0,\omega) = i\omega\mu_0 a I(\omega)\times \int_0^\infty \frac{\lambda}{m_g+m_0}J_1(\lambda a)J_1(\lambda r)e^{-m_0|h|}d\lambda.$$

Transient electric fields can be determined by taking Fourier transform to $E(r,z,\omega)$ as

$$E(r,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(r,z,\omega)e^{(-i\omega t)}d\omega.$$

Consequently, equation (19) can be written as

$$E_{air}(r,z,t) = \frac{\mu_0 a}{2}\int_0^\infty \lambda[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{i\omega I(\omega)}{m_0}[e^{-m_0|z+h|} - e^{m_0(z-|h|)}]\ e^{(-i\omega t)}d\omega]J_1(\lambda a)J_1(\lambda r)d\lambda + \mu_0 a\int_0^\infty \lambda[\frac{1}{2\pi}\int_{-\infty}^{\infty}[\frac{i\omega I(\omega)}{m_g+m_0}\times e^{m_0(z-|h|)}]e^{(-i\omega t)}d\omega]J_1(\lambda a)J_1(\lambda r)d\lambda$$

At this point, we consider the current switch-off by taking $t = 0$. We find that

$$I(\omega) = \int_{-\infty}^{\infty} I_0[1-u(t)]e^{(-i\omega t)}dt,$$

where I_0 is the amplitude of current, and $u(t)$ is step-function of Heaviside. Generally, we can regard ω as a complex variable, defined by $\omega_0 + i\kappa$ where κ is a small imaginary part. Taking Fourier transform to Heaviside step-function and the result is

$$U(\omega)=\frac{1}{i\omega}.$$

Since ω is real, we take $\kappa=0$. Consequently, $i\omega I(\omega)=I_0$. As $\mu_0\varepsilon_0$ is a very small positive number, then m_0 is approximately equal to λ. The result becomes

$$E_{air}(r,z,t)=\frac{\mu_0 a I_0}{2}\int_0^\infty [e^{-\lambda|z+h|}-e^{\lambda(z-|h|)}]\times \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(-i\omega t)}d\omega\right] J_1(\lambda a)J_1(\lambda r)d\lambda$$
$$+\mu_0 a I_0\int_0^\infty \lambda\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{m_g+\lambda}e^{(-i\omega t)}d\omega\right]\times e^{\lambda(z-|h|)}J_1(\lambda a)J_1(\lambda r)d\lambda.$$

Knowing that $e^{(-i\omega t)}$ is an analytic function, then $\int_{-\infty}^{\infty} e^{(-i\omega t)}d\omega=0$ (Wunsch, 1994). Thus,

$$E_{air}(r,z,t)=\mu_0 a I_0\int_0^\infty \lambda F_3 e^{\lambda(z-|h|)}\times J_1(\lambda a)J_1(\lambda r)d\lambda, \quad (21)$$

where $F_3=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{m_g+\lambda}e^{-i\omega t}d\omega$ The mathematical expression for F_3 may be determined by using contour integration. Applying the standard Bromwich contour to equation (21), then we obtain

$$E_{air}(r,z,t)=-\frac{aI_0}{\sigma_0}\int_0^\infty \lambda^2 e^{\lambda(z-|h|-\tau_3)}\times J_1(\lambda a)J_1(\lambda r)d\lambda,$$

where $\tau_3=\frac{t}{\mu_0\sigma_0}$ is a dimensionless time variable. Let $\varsigma_3=|h|-z$, thus

$$E_{air}(r,z,t)=-\frac{aI_0}{\sigma_0}\int_0^\infty \lambda^2 e^{-\lambda(\varsigma_3+\tau_3)}\times J_1(\lambda a)J_1(\lambda r)d\lambda. \quad (22)$$

As we can consider the coil of transmitter to be small ($a\to 0$), then we can approximate $J_1(\lambda a)$ by $\frac{\lambda a}{2}$. Thus,

$$E_{air}(r,z,t)=-\frac{a^2 I_0}{2\sigma_0}\times \int_0^\infty \lambda^3 e^{-\lambda(\varsigma_3+\tau_3)}J_1(\lambda r)d\lambda. \quad (23)$$

When $r\to 0$, that is, the distance between the transmitter and receiver is smaller, we have $J_1(\lambda r)\cong\frac{\lambda r}{2}$ and the transient electric fields now become

$$E_{air}(r,z,t)=-\frac{6ra^2 I_0}{\sigma_0(\varsigma_3+\tau_3)^5}. \quad (24)$$

Two Layered Earth Model with an Exponentially Increasing Conductivity Profile in Overburden

We now consider the ground in the form of two layers which represent to the area near sea shore. An overburden has conductivity given by

$$\sigma_{ove}(z)=\sigma_0 e^{-b(z-d)},\ 0\le z\le d,$$

where b and σ_0 are positive constants. The conductivity of host medium is a constant, which can be defined as

$$\sigma_{host}(z)=\sigma_0,\quad z\ge d.$$

At this point, we consider a primary alternating current source carried by a coil of radius a, at height $z=-h$ above the surface of the earth, $z=0$ (see Figure 2). We follow the same method as in the previous section and after doing the mathematical analysis, we obtain the transient electric fields

$$E_{air}(r,z,t)=\frac{3\rho\rho'^2 I_0 b^2}{2\sigma_0\zeta_1(\varsigma_4-\tau_4')^5}-\frac{3\rho\rho'^2 I_0 b^2}{8\sigma_0\zeta_1(1-\zeta_1)(\varsigma_4-\tau_4')^4}, \quad (25)$$

where $\rho=\frac{rb}{2}$, $\rho'=\frac{ab}{2}$, $\varsigma_1=e^{\frac{bd}{2}}$,

$\varsigma_4=\frac{b(|h|-z)}{2}$, $\tau_4'=\frac{\tau_4}{\varsigma_1(1-\varsigma_1)}$ and $\tau_4=\frac{tb^2}{\mu_0\sigma_0}$.

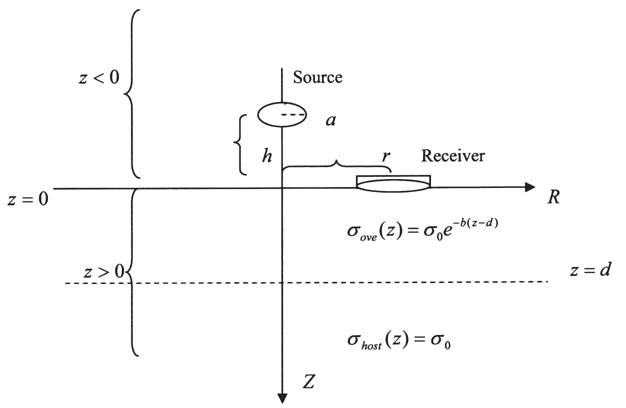

Figure 2 Illustration of two layered Earth model with an exponentially increasing conductivity profile in overburden

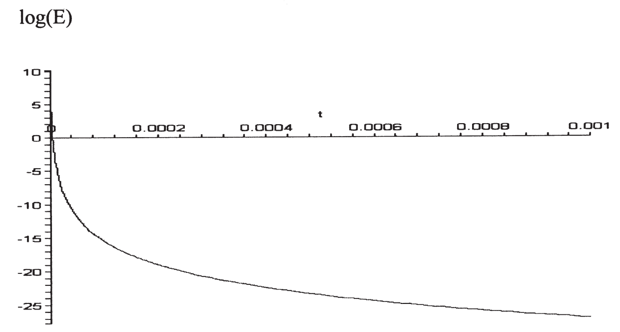

Figure 3 Graph of log *E* versus *t* for the half-space model with constant conductivity ground profile $\sigma_0 = 1$ Sm^{-1}

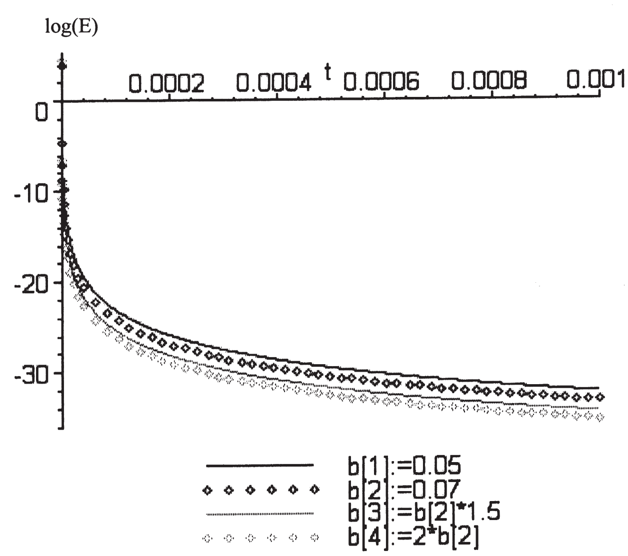

Figure 4 Graph of log *E* versus *t* for the half-space model with an exponentially increasing conductivity ground profile, $d = 10\ m$, $b_1 = 0.05$ m^{-1}, $b_2 = 0.07$ m^{-1}, $b_3 = 0.5b_2$ m^{-1}, $b_4 = 2b_2$ m^{-1} and $\sigma_0 = 1$ Sm^{-1}

Conclusions and Future Work

The objective of this paper is to present some mathematical models and techniques needed for studying the structures of the earth's surface layers. We briefly described the exploration of geomathematics and concentrated on the time-domain electromagnetic method. We formulated the problem to get the electric fields, which could be used to find the electric fields on the ground surface. We considered two models. The first one is for the ground having constant conductivity profile, defined by $\sigma_p(z) = \sigma_0$, where σ_0 is a positive constant. The second one is for the ground having two layers, the overburden and the host medium. The conductivity of overburden is given by $\sigma_{ove}(z) = \sigma_0 e^{-b(z-d)}, 0 \le z \le d$, where b is also positive constant. The conductivity of host medium, $z \ge d$, is a constant given by $\sigma_{host}(z) = \sigma_0$. The electric fields in each model were expressed in terms of mathematical expressions and plotted. The curves of transient electric fields of equations (24) and (25) decayed rapidly as shown in figures 3, 4 and 5. In case of an overburden with thickness 2 m, the curve of transient electric fields of equation (25) also decayed as shown in figure 5, but it was higher than the results shown in figure 4 in which the overburden thickness is 10 m. This result is caused by the thickness of conductive overburden layered. As the overburden thickness decreases, the transient electric fields of equation (25) tend to the transient electric fields generated by equation (24). These are shown in figures 3 and 5. The curves of transient electric fields of equations (24) and (25) decayed rapidly as we expected. The master curve of transient electric fields can be used to compare with the field measurement for describing the ground

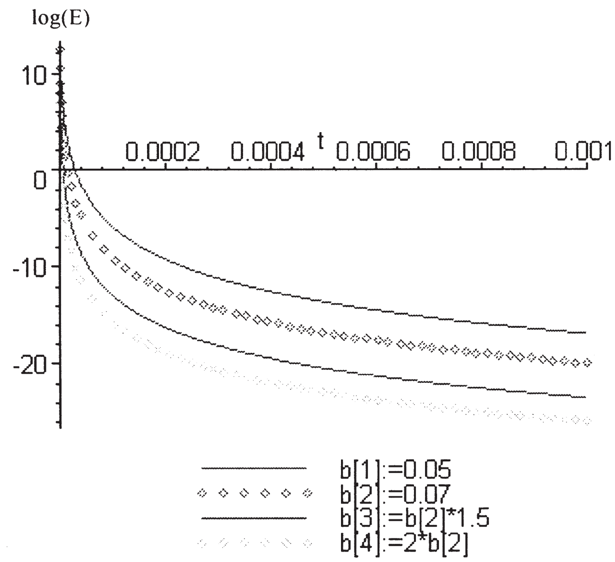

Figure 5 Graph of log E versus t for two layered Earth model with an exponentially increasing conductivity profile in overburden, $d = 2$ m, $b_1 = 0.05$ m^{-1}, $b_2 = 0.07$ m^{-1}, $b_3 = 1.5b_2$ m^{-1}, $b_4 = 2b_2$ m^{-1}, and $\sigma_0 = 1$ Sm^{-1}

structure. However, the equations (24) and (25) can be used for only the case $r \to 0$ and $a \to 0$, and these may cause the problem in practice to set the transmitter and receiver spacing.

It is possible that additional studies with more complicated layered structures can be conducted and new problems should also be solved by means of formulated equations in future work. Moreover, other mathematical methods may be applied to study such problem as well as those found in the future. Furthermore, it is also possible to use numerical methods in the future work to compare the results with those obtained from analytical ones.

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