Explicit Inverse of a Doubly Companion Matrix

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Abstract

Butcher and Chartier in first introduced a doubly companion matrix, after that Butcher and Wright used doubly companion matrices as a tool to analyze numerical methods and some general linear methods property. Explicit formula for a determinant and an inverse formula of the doubly companion matrix were proved.

Keywords: companion matrix, doubly companion matrix, inverse matrix

1. Introduction and Preliminaries

Let $\mathbb{C}$ be the field of complex numbers. For a positive integer n, let M_n be the set of all $n \times n$ matrices over $\mathbb{C}$, and let $I_n \in M_n$ be the identity matrix. The set of all complex vectors, or $n \times 1$ matrices over $\mathbb{C}$ is denoted by $\mathbb{C}^n$, and let $\bar{0}$ be the zero vector in $\mathbb{C}^n$. Doubly companion matrices $C \in M_n$ first introduced by Butcher and Chartier (1999), given by

$$C = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n - \beta_n \\ 1 & 0 & \cdots & 0 & -\beta_{n-1} \\ 0 & 1 & \dots & 0 & -\beta_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -\beta_2 \\ 0 & 0 & \cdots & 1 & -\beta_1 \end{bmatrix},$$

that is, a $n \times n$ matrix C with $n > 1$ is called a *(upper) doubly companion matrix* if its entries c_{ij} satisfy $c_{ij} = 1$ or all entries in the submain-diagonal of C and else $c_{ij} = 0$ for $i \neq 1$ and $j \neq n$. Butcher and Wright in used the doubly companion matrices as a tool for analyzing various extension of classical methods with inherent Runge-Kutta stability (Butcher and Chartier, 1999). The doubly companion matrices is important for application in some certain matrix equations, numerical and linear methods. In the present paper we give an explicit inverse formula for the doubly companion matrix.

2. Determinant of a Doubly Companion Matrix

Let $\alpha(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ and $\beta(x) = x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$ be two monic polynomials over complex numbers, we prefer to define the corresponding upper doubly companion matrix $U(\alpha, \beta)$ of $\alpha(x)$ and $\beta(x)$, and for convenience, we denote the upper doubly companion matrix by $U := U(\alpha, \beta)$, that is

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$$U=\begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0-b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}. \quad (2.1)$$

The lower doubly companion matrix is define by $L:=L(\alpha,\beta)$, that is

$$L=\begin{bmatrix} -a_{n-1} & 1 & \cdots & 0 & 0 \\ -a_{n-2} & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -a_1 & 0 & \cdots & 0 & 1 \\ -a_0-b_0 & -b_1 & \cdots & -b_{n-2} & -b_{n-1} \end{bmatrix}.$$

In this paper, the term doubly companion matrix refers to the upper doubly companion matrix or lower doubly companion matrix.

Now, we can be written the matrix U in a partitioned form as

$$U=\begin{bmatrix} -q^T & -b_0-a_0 \\ I_{n-1} & -p \end{bmatrix}_{(n,n)},$$

where $p=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}$ and $q=\begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix}$.

The author presented in computed the characteristic polynomial of the upper doubly companion matrix and proved that it is a nonderogatory matrix, that is, its characteristic polynomial is equal to its minimum polynomial, so that, we got the determinant formula from of the matrix from the constant terms of the characteristic polynomial, since the constant term is $(-1)^n$ times its determinant, where n is the size of the matrix. However, the determinant of the upper doubly companion matrix can direct computing as in following lemma (Wanicharpichat, 2011)

Lemma 2.1 *Let* $U=\begin{bmatrix} -q^T & -b_0-a_0 \\ I_{n-1} & -p \end{bmatrix}_{(n,n)}$*, be an upper doubly companion matrix, where* $p=\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}^T$*, and* $q=\begin{bmatrix} b_{n-1} & b_{n-2} & \ldots & b_1 \end{bmatrix}^T$*, then*

$$\det U=(-1)^n\left(a_0+b_0+\sum_{i+j=n} a_i b_j\right).$$

Proof. Consider the determinant of the doubly companion matrix

$$\det U=\begin{vmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0-b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{vmatrix}$$

by interchange the first row with each succeeding row until it is last, we have

$$\det U=(-1)^{n-1}\begin{vmatrix} 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & -a_{n-1} \\ -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0-b_0 \end{vmatrix},$$

adding the multiple of the first row by b_{n-1} to the last row, similarly adding the multiple of the second row by b_{n-2} to the last row, and so on, we have

$$\det U=(-1)^{n-1}\begin{vmatrix} 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \ddots & 1 & -a_{n-1} \\ 0 & 0 & \cdots & 0 & -a_0-b_0-\sum_{i+j=n} a_i b_j \end{vmatrix}$$

$$=(-1)^n\left(a_0+b_0+\sum_{i+j=n} a_i b_j\right).$$

This completes the proof.

3. Main Results

In fact, the inverse of nonsingular companion matrix is again in companion matrix form. But the inverse of the nonsingular doubly companion matrix is not a doubly companion matrix form.

Theorem 3.1 *Let* $U = \begin{bmatrix} -q^T & -b_0 - a_0 \\ I_{n-1} & -p \end{bmatrix}_{(n,n)}$, *be an upper doubly companion matrix, where* $p = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}^T$, *and* $q = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 \end{bmatrix}^T$, *and let* $\delta := \det U \neq 0,$ *then*

$$U^{-1} = \frac{1}{\delta} \begin{bmatrix} p & [a_i b_{n-i}]_{(n-1,n-1)} + \delta I_{n-1} \\ 1 & q^T \end{bmatrix}_{(n,n)}.$$

Proof. From Lemma 2.1, let

$$\det U = (-1)^n \left(a_0 + b_0 + \sum_{i+j=n} a_i b_j \right) \neq 0.$$

Therefore U is a nonsingular matrix. We claim that

$$U^{-1} = \frac{1}{\delta} \begin{bmatrix} a_1 & a_1 b_{n-1} + \delta & a_1 b_{n-2} & \cdots & a_1 b_1 \\ a_2 & a_2 b_{n-1} & a_2 b_{n-2} + \delta & \cdots & a_2 b_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-1} b_{n-1} & a_{n-1} b_{n-2} & \cdots & a_{n-1} b_1 + \delta \\ 1 & b_{n-1} & b_{n-2} & \cdots & b_1 \end{bmatrix}.$$

We would like to prove that $UU^{-1} = I_n = U^{-1}U.$ Now, let $v = \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \dots & a_{n-1} & 1 \end{bmatrix}^T.$

The matrix U^{-1} can be written in the following form $U^{-1} = \frac{1}{\delta}\begin{bmatrix} v & b_{n-1}v & b_{n-2}v & \dots & b_1 v \end{bmatrix} + \begin{bmatrix} \overline{0} & e_1 & e_2 & \cdots & e_{n-1} \end{bmatrix},$

where $e_1, e_2, \dots, e_{n-1}, e_n$ are all standard basis vectors in $\mathbb{C}^n$, and $\overline{0}$ is the zero vector in $\mathbb{C}^n$. From Ipsen (2009), view UU^{-1} is a block row vector of matrix vector product. The column of UU^{-1} are matrix vector products of U with column of U^{-1},

$$\begin{aligned} UU^{-1} &= U\left(\frac{1}{\delta}\begin{bmatrix} v & b_{n-1}v & b_{n-2}v & \dots & b_1 v \end{bmatrix} + \begin{bmatrix} \overline{0} & e_1 & e_2 & \cdots & e_{n-1} \end{bmatrix} \right), \\ &= U\begin{bmatrix} \frac{1}{\delta}v & e_1 + \frac{b_{n-1}}{\delta}v & e_2 + \frac{b_{n-2}}{\delta}v & \dots & e_{n-1} + \frac{b_1}{\delta}v \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\delta}Uv & Ue_1 + \frac{b_{n-1}}{\delta}Uv & Ue_2 + \frac{b_{n-2}}{\delta}Uv & \dots & Ue_{n-1} + \frac{b_1}{\delta}Uv \end{bmatrix}. \end{aligned} \tag{3.1}$$

But

$$Uv = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0 - b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ 1 \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = \delta e_1.$$

Hence, equation (3.1) become,

$$UU^{-1} = \begin{bmatrix} e_1 & Ue_1 + b_{n-1}e_1 & Ue_2 + b_{n-2}e_1 & \dots & Ue_{n-1} + b_1 e_1 \end{bmatrix}. \tag{3.2}$$

Now consider:

$$Ue_1 = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0-b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = -b_{n-1}e_1 + e_2,$$

$$Ue_2 = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0-b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = -b_{n-2}e_1 + e_3,$$

$$\vdots$$

$$Ue_{n-1} = \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0-b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} = -b_1e_1 + e_n.$$

Substituting $Ue_1, Ue_2, \ldots, Ue_{n-1}$ from above into equation (3.2). We have

$$UU^{-1} = \begin{bmatrix} e_1 & e_2 & e_3 & \ldots & e_n \end{bmatrix} = I_n.$$

Finally, we shall show that $U^{-1}U = I_n$. From our hypothesis

$$U^{-1} = \frac{1}{\delta} \begin{bmatrix} a_1 & a_1b_{n-1}+\delta & a_1b_{n-2} & \cdots & a_1b_1 \\ a_2 & a_2b_{n-1} & a_2b_{n-2}+\delta & \cdots & a_2b_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-1}b_{n-1} & a_{n-1}b_{n-2} & \cdots & a_{n-1}b_1+\delta \\ 1 & b_{n-1} & b_{n-2} & \cdots & b_1 \end{bmatrix}.$$

Now, let $r = \begin{bmatrix} 1 & b_{n-1} & b_{n-2} & \cdots & b_1 \end{bmatrix}$ be a row vector, we write

$$U^{-1} = \frac{1}{\delta} \begin{bmatrix} a_1r + \delta e_2^T \\ a_2r + \delta e_3^T \\ \vdots \\ a_{n-1}r + \delta e_n^T \\ r \end{bmatrix}.$$

From Ipsen and Ilse (2009) again, view $U^{-1}U$ as a block column vector of matrix vector products, where the rows of $U^{-1}U$ are matrix vector products of the rows of U^{-1} with

$$U^{-1}U = \frac{1}{\delta} \begin{bmatrix} a_1r + \delta e_2^T \\ a_2r + \delta e_3^T \\ \vdots \\ a_{n-1}r + \delta e_n^T \\ r \end{bmatrix} U = \frac{1}{\delta} \begin{bmatrix} a_1rU + \delta e_2^TU \\ a_2rU + \delta e_3^TU \\ \vdots \\ a_{n-1}rU + \delta e_n^TU \\ rU \end{bmatrix}. \quad (3.3)$$

Consider:

$$rU = \begin{bmatrix} 1 & b_{n-1} & b_{n-2} & \ldots & b_1 \end{bmatrix} \times \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0-b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & \ldots & 0 & \delta \end{bmatrix}$$
$$= \delta e_n^T.$$

From equation (3.3), we have

$$U^{-1}U = \frac{1}{\delta} \begin{bmatrix} a_1\delta e_n^T + \delta e_2^TU \\ a_2\delta e_n^T + \delta e_3^TU \\ \vdots \\ a_{n-1}\delta e_n^T + \delta e_n^TU \\ \delta e_n^T \end{bmatrix}. \quad (3.4)$$

Consider:

$$e_2^TU = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0-b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} = e_1^T - a_1e_n^T$$

$$e_3^T U = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \end{bmatrix} \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0 - b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} = e_2^T - a_2 e_n^T,$$

$$\vdots$$

$$e_n^T U = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} -b_{n-1} & -b_{n-2} & \cdots & -b_1 & -a_0 - b_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} = e_{n-1}^T - a_{n-1} e_n^T.$$

Replacing above equations in to (3.4), we have

$$U^{-1}U = \frac{1}{\delta} \begin{bmatrix} a_1 \delta e_n^T + \delta(e_1^T - a_1 e_n^T) \\ a_2 \delta e_n^T + \delta(e_2^T - a_2 e_n^T) \\ \vdots \\ a_{n-1} \delta e_n^T + \delta(e_{n-1}^T - a_{n-1} e_n^T) \\ \delta e_n^T \end{bmatrix} = \frac{1}{\delta} \begin{bmatrix} \delta e_1^T \\ \delta e_2^T \\ \vdots \\ \delta e_{n-1}^T \\ \delta e_n^T \end{bmatrix} = I_n \cdot$$

This completes the proof.

Theorem 3.2

Let $L(\alpha,\beta) = \begin{bmatrix} -\tilde{p} & I_{n-1} \\ -b_0 - a_0 & -\tilde{q}^T \end{bmatrix}_{(n,n)}$, *be a lower doubly companion matrix, where*

$\tilde{p} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 \end{bmatrix}^T$, *and*

$\tilde{q} = \begin{bmatrix} b_1 & b_2 & \dots & b_{n-1} \end{bmatrix}^T$, *and let*

$\delta := \det L(\alpha,\beta) \neq 0$, *then*

$$L(\alpha,\beta)^{-1} = \frac{1}{\delta} \begin{bmatrix} \tilde{q}^T & 1 \\ [a_{n-i} b_i]_{(n-1,n-1)} + \delta I_{n-1} & \tilde{p} \end{bmatrix}_{(n,n)}.$$

Proof. Since the upper doubly companion matrix $U(\alpha,\beta)$ similar to the lower doubly companion matrix $L(\alpha,\beta)$ via the *backward identity* matrix of order $n \times n$ (or *reversal* matrix of order $n \times n$), $J\ (= J^{-1})$ (Horn and Johnson, 1996), where

$$J = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}_{(n,n)}$$

that is, $L(\alpha,\beta) = J^{-1} U(\alpha,\beta) J$.

Therefore the determinant of $L(\alpha,\beta)$ is the same as that of $U(\alpha,\beta)$, we have

$$\det L(\alpha,\beta) = (-1)^n \left(a_0 + b_0 + \sum_{i+j=n} a_i b_j \right).$$

If $\det L(\alpha,\beta) \neq 0$ then the matrix $L(\alpha,\beta)$ is nonsingular and by Theorem 3.1, we have

$$\begin{aligned} L(\alpha,\beta)^{-1} &= J^{-1} U(\alpha,\beta)^{-1} J \\ &= \frac{1}{\delta} J^{-1} \begin{bmatrix} p & [a_i b_{n-i}]_{(n-1,n-1)} + \delta I_{n-1} \\ 1 & q^T \end{bmatrix}_{(n,n)} J \\ &= \frac{1}{\delta} \begin{bmatrix} \tilde{q}^T & 1 \\ [a_{n-i} b_i]_{(n-1,n-1)} + \delta I_{n-1} & \tilde{p} \end{bmatrix}_{(n,n)}, \end{aligned}$$

where $\tilde{p}=\begin{bmatrix}a_{n-1} & a_{n-2} & \cdots & a_1\end{bmatrix}^T$, and $\tilde{q}=\begin{bmatrix}b_1 & b_2 & \ldots & b_{n-1}\end{bmatrix}^T$.

In particular, there are four special cases of the doubly companion, namely the companion matrices. Now, by Theorem 3.1 and Theorem 3.2 we have the well known explicit formulas of the inverses as in the following four corollaries.

Corollary 3.3 *Let* $U(\alpha)=\begin{bmatrix}\bar{0}^T & -a_0 \\ I_{n-1} & -p\end{bmatrix}_{(n,n)}$, *be an upper companion matrix, where* $p=\begin{bmatrix}a_1 & a_2 & \cdots & a_{n-1}\end{bmatrix}^T$, *and let* $\delta := \det U(\alpha) = (-1)^n a_0 \neq 0$, *then*

$$U(\alpha)^{-1}=\frac{1}{\delta}\begin{bmatrix}p & \delta I_{n-1} \\ 1 & \bar{0}^T\end{bmatrix}_{(n,n)}.$$

Corollary 3.4 *Let* $U(\beta)=\begin{bmatrix}-q^T & -b_0 \\ I_{n-1} & \bar{0}\end{bmatrix}_{(n,n)}$, *be an upper doubly companion matrix, where* $q=\begin{bmatrix}b_{n-1} & b_{n-2} & \ldots & b_1\end{bmatrix}^T$, *and let* $\delta := \det U(\beta) = (-1)^n b_0 \neq 0$, *then*

$$U(\beta)^{-1}=\frac{1}{\delta}\begin{bmatrix}p & \delta I_{n-1} \\ 1 & q^T\end{bmatrix}_{(n,n)}.$$

Corollary 3.5 *Let* $L(\alpha)=\begin{bmatrix}-\tilde{p} & I_{n-1} \\ -a_0 & \bar{0}^T\end{bmatrix}_{(n,n)}$, *be a lower doubly companion matrix, where* $\tilde{p}=\begin{bmatrix}a_{n-1} & a_{n-2} & \cdots & a_1\end{bmatrix}^T$ *and let* $\delta := \det L(\alpha) = (-1)^n a_0 \neq 0$, *then*

$$L(\alpha)^{-1}=\frac{1}{\delta}\begin{bmatrix}\bar{0}^T & 1 \\ \delta I_{n-1} & \tilde{p}\end{bmatrix}_{(n,n)}.$$

Corollary 3.6 *Let* $L(\beta)=\begin{bmatrix}\bar{0} & I_{n-1} \\ -b_0 & \tilde{q}^T\end{bmatrix}_{(n,n)}$, *be a lower doubly companion matrix, where* $\tilde{q}=\begin{bmatrix}b_1 & b_2 & \ldots & b_{n-1}\end{bmatrix}^T$, *and let* $\delta := \det L(\beta) = (-1)^n b_0 \neq 0$, *then*

$$L(\beta)^{-1}=\frac{1}{\delta}\begin{bmatrix}\tilde{q}^T & 1 \\ \delta I_{n-1} & \bar{0}\end{bmatrix}_{(n,n)}.$$

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