On Semipolygroups

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Abstract

In this paper, we introduce the notions of semipolygroups and (strongly) regular relation, and investigate their basic properties. Moreover, we review the concepts of polygroups and some of their properties are also described.

Keywords: semipolygroup; (strongly) regular relation; polygroup

1. Introduction

Marty (1934) first presented the concept of hyperstructure. It is well known that the class of hyperstructures is generalized from group theory. In a group, the combination between two elements (of a non-empty set) is an element but in a hypergroup, the combination between two elements is a set. The semihypergroups are an associative property of hypergroups which are based on the concept of hyperoperation. Later, Corsini (1993) introduced the fundamental theory of hyperstructures and many notion of hyperstructure can be found in his work. Next, the special subclasses of hypergroup called polygroups were studied by Comer (1996). He studied polygroups and applied hyperstructures with algebras and color schemes. Other researchers developed the concept of hyperstructure, for example, Davvaz (2000) proved new identities of strong regularity and fuzzy strong regularity on semihypergroups, and presented results on congruences of semihypergroups. Moreover, Davvaz (2010) considered the normal subpolygroups and homomorphisms between polygroups and identified the isomorphism theorems of polygroups. Furthermore, Jafarabadi *et al*. (2012) introduced new kinds of hyperstructure called simple and completely simple semihypergroups, and presented methods for constructing these new classes of hyperstructure and considered the regularity of semihypergroups and Davvaz (2013), later, discussed polygroup theory and related systems.

The purpose of this paper is to examine the notions of semipolygroups and investigate their basic properties. Furthermore, we introduce the morphism on semipolygroups and the (strongly) regular equivalence relation on semipolygroups. Moreover, we review the concepts of polygroups and some their properties are also described.

2. Semipolygroups

In this section, we recall the prelimnary

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definitions and their necessary results. Furthermore, some results on semipolygroups are presented.

Definition 2.1: Let S be a non-empty set and let $P^*(S)$ be the set of all non-empty subsets of S. A *hyperoperation* on S is a map $\circ : S \times S \to P^*(S)$. A *polygroupoid* is a system $(S,\circ)$, where $\circ$ is a hyperoperation, i.e., $\emptyset \neq x \circ y = \circ(x,y) \subseteq S$ for all x, y of S. If A and B are non-empty subsets of S, then we denote

$$A \circ B = \bigcup_{a \in A,\ b \in B} a \circ b,$$

$$x \circ A = \{x\} \circ A \text{ and } A \circ x = A \circ \{x\}.$$

A polygroupoid $(S,\circ)$ is called a *semipolygroup* if, for all x, y, z of S, we have $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u \in x \circ y} u \circ z = \bigcup_{v \in y \circ z} x \circ v.$$

If a semipolygroup $(S,\circ)$ has the property that, $x \circ y = y \circ x$ for all x, y in S, we shall say that S is a *commutative semipolygroup*. If a semipolygroup $(S,\circ)$ contains an element e with the property that, for all x in S,

$$e \circ x = x \circ e = \{x\}.$$

We say that e is an *identity element* of S, and that S is a *semipolygroup with identity*.

Example 2.2: Let $S = \{a, b, c\}$ with the following multiplication table:

$\cdot$	a	b	c
a	a	$\{a,b\}$	$\{a,c\}$
b	$\{a,b\}$	b	$\{b,c\}$
c	$\{a,c\}$	$\{b,c\}$	c

Then, $(S,\cdot)$ is a semipolygroup.

Example 2.3: Let $(\mathbb{Z}, +)$ be a group. Define a hyperoperation $\circ$ on $\mathbb{Z}$ by

$$x \circ y = \{x + y,\ x - y\} \text{ for all } x, y \text{ in } \mathbb{Z}.$$

Then, $(\mathbb{Z},\circ)$ is a semipolygroup.

Definition 2.4: Let $(S,\circ)$ be a semipolygroup with at least two elements and 0 be an element of S. If $0 \circ x = \{0\}$ for all x in S, then 0 is called a *left zero element* of S, and S is called a *semipolygroup with left zero*. Similarly, 0 is called a *right zero element* of S, and that S is a *semipolygroup with right zero* if $x \circ 0 = \{0\}$ for all x in S. We say that 0 is a *zero element* (or just a *zero*) of S if it is both a left and a right zero element of S, and that S is a *semipoly-group with zero* if it is both a semipolygroup with left and right zero.

Example 2.5: Let S be a non-empty set with at least two elements. If we define a hyperoperation $\circ$ on S by

$$x \circ y = \{x\} \text{ for all } x, y \text{ in } S,$$

then S is a semipolygroup. For any $a \in S$, $a \circ x = \{a\}$ for all $x \in S$, so a is a left zero element of S and S is called a *left zero semipolygroup*. Also, if we define a hyperoperation $*$ on S by

$$x * y = \{y\} \text{ for all } x, y \text{ in } S,$$

then S is a semipolygroup. For any $b \in S$, $x * b = \{b\}$ for all $x \in S$, so b is a right zero element of S and S is called a *right zero semipolygroup*. For x, y in a semipolygroup $(S,\circ)$, we write the product of x, y as xy instead of $x \circ y$.

Definition 2.6: Let S be a semipolygroup. For all $a \in S$ and $k \in \mathbb{N}$, $a^1 = a$ and $a^{k+1} = a^k a$. Then, we have the following lemma.

Lemma 2.7 Let S be a semipoly-group. For all $a \in S$ and $m, n \in \mathbb{N}$,

(i) $a^n = a^{m+n}$;

(ii) $(a^m)^n = a^{mn}$.

Proof. (i) Let $P(n)$ be a statement $a^m a^n = a^{m+n}$ for all $m \in \mathbb{N}$. If $n = 1$, since $a^m a^1 = a^{m+1}$ for all $m \in \mathbb{N}$, then $P(1)$ is true. Suppose that $P(k)$ is true, where $k \in \mathbb{N}$. Then, $a^m a^k = a^{m+k}$ for all $m \in \mathbb{N}$. Therefore, $a^m a^{k+1} = a^m(a^k a) = (a^m a^k)a = a^{m+k}a = a^{(m+k)+1} = a^{m+(k+1)}$ for all $m \in \mathbb{N}$. Hence, $P(k+1)$ is true. By the mathematical induction, we have that $P(n)$ is true for all $n \in \mathbb{N}$.

(ii) Let $P(n)$ be a statement $(a^m)^n = a^{mn}$ for all $m \in \mathbb{N}$. If $n = 1$, since $(a^m)^1 = a^m = a^{m1}$ for all $m \in \mathbb{N}$, then $P(1)$ is true. Suppose that $P(k)$ is true, where $k \in \mathbb{N}$. Then, $(a^m)^k = a^{mk}$ for all $m \in \mathbb{N}$. Therefore, using (i), $(a^m)^{k+1} = (a^m)^k a^m = a^{mk} a^m = a^{mk+m} = a^{m(k+1)}$ for all $m \in \mathbb{N}$. Hence, $P(k+1)$ is true. By the mathematical induction, we have that $P(n)$ is true for all $n \in \mathbb{N}$. ■

Definition 2.8: An element S in a semipolygroup S is called an *idempotent* if $s \in s^2$ and the element $a \in S$ is called a *scalar* if

$$|ax| = |xa| = 1 \text{ for all } x \text{ in } S,$$

where $|A|$ is the number of elements in a set A.

Definition 2.9: A non-empty subset T of a semipolygroup S is called a *subsemipolygroup* if $xy \subseteq T$ for all $x, y \in T$.

Example 2.10: Let S be a semipolygroup and $s \in S$ be a scalar idempotent, that is $s^2 = ss = \{s\}$. Then, $\{s\}$ is a subsemipolygroup of S.

Definition 2.11: A semipolygroup S is called a *rectangular band* if $aba = \{a\}$ for all a, b in S.

Definition 2.12: A map $\phi: S \to T$, where $(S, \circ)$ and $(T, *)$ are semipolygroups, is called a *morphism* (or *homomorphism*) if, for all x, y in S,

$$\phi(x \circ y) = \phi(x) * \phi(y)$$
$$\text{or } \phi(xy) = \phi(x)\phi(y).$$

If ϕ is an injection, we shall call it a *monomorphism*. A morphism ϕ is called an *isomorphism* if it is a bijection. If there exists an isomorphism $\phi: S \to T$ we say that S and T are *isomorphic*, and write $S \cong T$. Then, we have the following results.

Proposition 2.13: Let $\phi: S \to T$ be a morphism, where S and T are left (or right) semipolygroups. Suppose that for all semipolygroup U and for all morphisms $\alpha, \beta : U \to S$,

$$\phi \circ \alpha = \phi \circ \beta \text{ implies } \alpha = \beta.$$

Then, ϕ is a monomorphism.

Proof. Let $\phi(x_1) = \phi(x_2)$, where $x_1, x_2 \in S$. Let $U = \{p\}$ be a singleton set such that $pp = \{p\}$ and let $\alpha, \beta : U \to S$ be defined by $\alpha(p) = x_1$ and $\beta(p) = x_2$. Then, U is a semipolygroup and α, β are morphisms. It follows that $\phi \circ \alpha = \phi \circ \beta$. Thus, $\alpha = \beta$ implies $x_1 = x_2$. Therefore, ϕ is a monomorphism. ■

Lemma 2.14: Let $\phi: S \to T$ be a morphism, where S and T are semipolygroups.

(i) If S is an idempotent of S, then $\phi(s)$ is an idempotent of T.

(ii) If a is a scalar idempotent of S, then $\phi(a)$ is a scalar idempotent of T.

(iii) $\phi(S)$ is a semipolygroup.

Proof. (i) Suppose that s is an idempotent of S. Then $s \in s^2$. Since $\phi(s) \in \phi(s^2) = \phi(s)$

$\phi(s) = \phi(s)^2$, $\phi(s)$ is an idempotent of T.

(ii) Suppose that a is a scalar idempotent of S. By (1), $\phi(a)$ is an idempotent of T. Since $a^2 = aa = \{a\}$, we have that $\phi(a)\phi(a) = \phi(aa) = \phi(\{a\}) = \{\phi(a)\}$. Therefore, $\phi(a)$ is a scalar idempotent of T.

(iii) Let $\phi(a), \phi(b), \phi(c) \in \phi(S)$. Then, $a, b, c \in S$. We see that $ab \subseteq S$. Let $z \in \phi(a)\phi(b) = \phi(ab)$. Then $z = \phi(x)$ for some $x \in ab \subseteq S$. Thus, $z = \phi(x) \in \phi(ab) \subseteq \phi(S)$ and hence, $\phi(a)\phi(b) \subseteq \phi(S)$.

Let $\in \big(\phi(a)\phi(b)\big)\phi(c) = \phi(ab)\phi(c)$. Then, $y \in \phi(z)\phi(c) = \phi(zc)$ for some $z \in ab \subseteq S$. That is $y = \phi(m)$ for some $m \in zc \subseteq (ab)c = a(bc)$. Thus, there exists $n \in bc$ such that $m \in an$. It implies that $= \phi(m) \in \phi(an) = \phi(a)\phi(n) \subseteq \phi(a)\phi(bc) = \phi(a)\big(\phi(b)\phi(c)\big)$. Therefore, $\big(\phi(a)\phi(b)\big)\phi(c) \subseteq \phi(a)\big(\phi(b)\phi(c)\big)$ Similarly, let $u \in \phi(a)\big(\phi(b)\phi(c)\big) = \phi(a)\phi(bc)$. It follows that $u \in \phi(a)\phi(k) = \phi(ak)$ for some $k \in bc \subseteq S$. That is $u = \phi(p)$ for some $p \in ak \subseteq a(bc) = (ab)c$. Thus there exists $q \in ab$ such that $m \in qc$. It implies that $= \phi(p) \in \phi(qc) = \phi(q)\phi(c) \subseteq \phi(ab)\phi(c) = \big(\phi(a)\phi(b)\big)\phi(c)$. Therefore, $(a)\big(\phi(b)\phi(c)\big) \subseteq \big(\phi(a)\phi(b)\big)\phi(c)$. Consequently, $\phi(a)\big(\phi(b)\phi(c)\big) = \big(\phi(a)\phi(b)\big)\phi(c)$. Hence, $\phi(S)$ is a semipolygroup. ■

Lemma 2.15: Let $\phi: S \to T$ be a morphism, where S and T are semipolygroups.

(i) If S is a rectangular band, then $\phi(S)$ is a rectangular band.

(ii) If e is an identity element of S, then $\phi(e)$ is an identity element of $\phi(S)$.

(iii) If 0 is a left zero element of S, then $\phi(0)$ is a left zero element of $\phi(S)$.

(iv) If 0 is a right zero element of S, then $\phi(0)$ is a right zero element of $\phi(S)$.

(v) If 0 is a zero element of S, then $\phi(0)$ is a zero element of $\phi(S)$.

(vi) If S is a commutative semipolygroup, then $\phi(S)$ is a commutative semipolygroup.

Proof. Let $\phi: S \to T$ be a morphism, where S and T are semipolygroups. By Lemma 2.14 (iii), $\phi(S)$ is a semipolygroup.

(i) Suppose that S is a rectangular band. Let $\phi(x), \phi(y) \in \phi(S)$. Then, $x, y \in S$ and we get that $xyx = \{x\}$. It follows that $(x)\phi(y)\phi(x) = \phi(xyx) = \phi(\{x\}) = \{\phi(x)\}$, as required.

(ii) Let e be an identity element of S and $\phi(x) \in \phi(S)$. Then $\phi(e) \in \phi(S)$ and $x \in S$. It implies that $(e)\phi(x) = \phi(ex) = \phi(\{x\}) = \{\phi(x)\} = \phi(\{x\}) = \phi(xe) = \phi(x)\phi(e)$. Therefore, $\phi(e)$ is an identity element of $\phi(S)$. Assume that 0 is a left zero element of S.

(iii) Let $\phi(x) \in \phi(S)$. Then, $x \in S$ and hence, we have $0x = \{0\}$. Thus, $\phi(0)\phi(x) = \phi(0x) = \phi(\{0\}) = \{\phi(0)\}$ and then $\phi(0)$ is a left zero element of $\phi(S)$.

(iv) Suppose that 0 is a right zero element of S. Let $\phi(x) \in \phi(S)$. Then, $x \in S$ and so we have $0 = \{0\}$. Thus, $\phi(x)\phi(0) = \phi(x0) = \phi(\{0\}) = \{\phi(0)\}$ and then $\phi(0)$ is a right zero element of $\phi(S)$.

(v) If 0 is a zero element of S, then 0

is both a left and a right zero element of S. By (iii) and (iv), we have $\phi(0)$ is a zero element of $\phi(S)$.

(vi) Suppose that S is a commutative semipolygroup. Let $\phi(x), \phi(y) \in \phi(S)$. Then, $x, y \in S$ and thus, $\phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x)$. Hence, $\phi(S)$ is a commutative semipolygroup. ■

Proposition 2.16: (Jafarabadi *et al.*, 2012) If $(S,\cdot)$ and $(T,\circ)$ are semipolygroups, then the Cartesian product $S \times T$ becomes a semipoly-group if we define

$$(s,t) \diamond (s',t') = (s \cdot s') \times (t \circ t') = \bigcup_{x \in s \cdot s', y \in t \circ t'} \{(x,y)\}.$$

We refer to that semipolygroup as the direct product of S and T.

Example 2.17: Let S be a semipolygroup with at least two elements. If A is a left zero semipolygroup of S and B is a right zero semipolygroup of S, then $A \times B$ is a semipolygroup and $(a,b)(a',b') = \bigcup_{x \in aa', y \in bb'} \{(x,y)\} = \{(a,b')\}$ for all $(a,b), (a',b') \in A \times B$.

Theorem 2.18: Let S be a semipoly-group. Then, the following conditions are equivalent:

(i) S is a rectangular band;

(ii) every element of S is a scalar idempotent, and $abc = ac$ for all a, b, c in S;

(iii) there exist a left zero semipoly-group L and a right zero semipolygroup R such that $S \cong L \times R$;

(iv) S is isomorphic to a semipolygroup of the form $A \times B$, where A and B are non-empty sets, and the hyperoperation on $A \times B$ is given by $(a,b)(a',b') = \{(a,b')\}$ for all $(a,b), (a',b') \in A \times B$.

Proof. Let $a, b, c \in S$.

(i) $\Longrightarrow$ (ii). Suppose that S is a rectangular band. Then, $\{a\} = aa^2a = a^4 = a^3a = a^2$ and so a is a scalar idempotent. Since $\{a\} = aba, \{c\} = cbc$ and $\{b\} = b(ac)b$, we have that $ac = (aba)(cbc) = a(bacb)c = a\{b\}c = abc$, as required.

(ii) $\Longrightarrow$ (iii). Assume that every element of S is a scalar idempotent, and $abc = ac$ for all a, b, c in S. Let $L = Sc$ and $R = cS$, where $\boldsymbol{c}$ is a fix element of S. If $x \in L = Sc$, then $\{x\} = zc$ for some $z \in S$. For all $y \in L = Sc$, we have $\{y\} = wc$ for some $w \in S$ and hence, $xy = (zc)(wc) = z(cwc) = zc = \{x\}$. Thus, L is a left zero semipolygroup. Similarly, R is a right zero semipolygroup. If we define $\phi: L \rightarrow R$ by $\phi(x) = \bigcup_{a \in xc, b \in cx} \{(a,b)\}$ for all $x \in S$, we obtain that $S \cong L \times R$.

(iii) $\Longrightarrow$ (iv). Suppose that $S \cong L \times R$, where L is a left zero semipolygroup and R is a right zero semipolygroup. Take $A = L$ and $B = R$, we are done.

(iv) $\Longrightarrow$ (i). Suppose that $A \times B \cong S$, where A and B are non-empty sets, and the hyperoperation on $A \times B$ is given by $(a,b)(a',b') = \{(a,b')\}$ for all (a,b), $(a',b') \in A \times B$. Then, there exists an isomor-phism $\phi: A \times B \rightarrow S$. Let $x, y \in S$. Then, $x = \phi(a,b)$ and $y = \phi(a',b')$ for some (a,b), $(a',b') \in A \times B$. Since $= \phi(a,b)\phi(a',b')\phi(a,b) = \phi\big((a,b)(a',b')\big)\phi(a,b) =$

$\phi(\{(a,b')\})\phi(a,b) = \phi((a,b')(a,b)) = \phi(\{(a,b)\}) = \{\phi(a,b)\} = \{x\}$, S is a rectangular band. ■

Next, we introduce the notion of the (strongly) regular equivalence relation on semi-polygroups and investigate some of its properties.

Definition 2.19: Let $(S,\circ)$ be a semipolygroup and ρ be an equivalence relation on S. Then, $A\bar{\rho}B$ means that for all $a \in A$ there exists $b \in B$ such that $a\rho b$ and for all $b' \in B$ there exists $a' \in A$ such that $a'\rho b'$; $A\bar{\bar{\rho}}B$ means that $a\rho b$ for all $a \in A$ and $b \in B$, where A and B are non-empty subsets of S. (We see that if $A\bar{\bar{\rho}}B$, then $A\bar{\rho}B$.)

(i) ρ is called *regular on the left* (*on the right*) if, for all x, y, a of $S, x\rho y \Rightarrow (ax)\bar{\rho}(ay)$ $((xa)\bar{\rho}(ya)$ respectively$)$;

(ii) ρ is called *strongly regular on the left* (*on the right*) if, for all x, y, a of $S, x\rho y \Rightarrow (ax)\bar{\bar{\rho}}(ay)$ $((xa)\bar{\bar{\rho}}(ya)$ respectively$)$;

(iii) ρ is called *regular* (*strongly regular*) if it is regular (strongly regular) on the left and on the right.

Remark: If ρ is a strongly regular relation, then ρ is a regular relation.

Theorem 2.20: (Davvaz, 2013) Let $(S,\circ)$ be a semipolygroup and ρ be an equivalence relation on S.

(i) If ρ is a regular relation on S, then the quotient set S/ρ is a semipolygroup with respect to the following the hyperoperation: $\rho(x)\otimes\rho(y) = \{\rho(z): z \in xy\}$.

(ii) If the hyperoperation defined by (i) is well-efined on S/ρ, then ρ is regular.

Theorem 2.21: (Davvaz, 2000) Let S be a semipolygroup and let ρ be a regular on S. Then, S/ρ is a semipolygroup with respect to the hyperoperation defined by Theorem 2.20 (i) and the map $\rho^{\emptyset}$ from S onto S/ρ given by $\rho^{\emptyset}(x) = \rho(x)$ for all $x \in S$ is a morphism. Now, let T be a semipolygroup and let $\phi: S \to T$ be a morphism. Then, the relation $\ker\phi = \phi \circ \phi^{-1} = \{(a,b) \in S \times S: \phi(a) = \phi(b)\}$ is regular on S, and there is a monomorphism $\alpha: S/\ker\phi \to T$ such that $\mathrm{im}\,\alpha = \mathrm{im}\,\phi$.

Theorem 2.22: (Davvaz, 2000) Let ρ be a regular on a semipolygroup S and let $\phi: S \to T$ be a morphism such that $\rho \in \ker\phi$. Then, there is a unique morphism $\beta: S/\rho \to T$ such that $\mathrm{im}\,\beta = \mathrm{im}\,\phi$.

Theorem 2.23: Let ρ, σ be regular relations on a semipolygroup S sucht that $\rho \subseteq \sigma$. Then, $\sigma/\rho = \{(\rho(x), \rho(y)) \in (S/\rho) \times (S/\rho) : (x,y) \in \sigma\}$ is a regular on S/ρ and $(S/\rho)/(\sigma/\rho) \cong S/\sigma$.

Proof. The Theorem 2.21 implies that there is a morphism β from S/ρ onto S/σ such that β is given by $\beta(\rho(a)) = \sigma(a)$ for all $a \in S$, and the regular $\ker\beta$ on S/ρ is given by $\ker\beta = \{(\rho(x),\rho(y)) \in (S/\rho) \times (S/\rho) : \beta(\rho(x)) = \beta(\rho(y))\} = \{(\rho(x),\rho(y)) \in (S/\rho) \times (S/\rho) : \sigma(x) = \sigma(y)\} = \{(\rho(x),\rho(y)) \in (S/\rho) \times (S/\rho) : (x,y) \in \sigma\}$.

It is usual to write $\ker\beta$ as σ/ρ. From Theorem 2.21, it now follows that there is monomorphism $\alpha: (S/\rho)/(\sigma/\rho) \to S/\rho$ defined by $\alpha(\sigma/\rho(\rho(x))) = \sigma(x)$ for all $a \in S$. For all $\sigma(a) \in S/\sigma$, since $a \in S$, we have $\rho(a) \in S/\rho$, there is $\sigma/\rho(\rho(x)) \in (S/$

$\rho)/(\sigma/\rho)$ such that $\alpha(\sigma/\rho(\rho(x))) = \sigma(x)$ and α is surjective. Hence, α is an isomorphism. ■

Definition 2.24: (Jafarabadi *et al.*, 2012) An element a of a semipolygroup S is called *regular* if there exists x in S such that $a \in axa$. The semipolygroup S is called *regular* if all its elements are regular.

Theorem 2.25: Let $\phi: S \to T$ be a monomorphism from a regular semipolygroup S into a semipolygroup T. Then, $\text{im}\,\phi$ is regular. If f is an idempotent in $\text{im}\,\phi$, then there exists an idempotent e in S such that $\phi(e) = f$.

Proof. Let $(e) \in \text{im}\,\phi = \{\phi(s) : s \in S\}$. There exists x in S such that $s \in sxs$, because S is regular. Thus, $\phi(s) \in \phi(sxs) = \phi(s)\phi(x)\phi(s)$ and hence, $\text{im}\,\phi$ is regular. Suppose that f is an idempotent in im ϕ. Since $f \in \text{im}\,\phi$, there exists $e \in S$ such that $\phi(e) = f$. We have $\phi(e) = f \in ff = \phi(e)\phi(e) = \phi(ee)$, that is $\phi(e) \in \phi(ee) = \bigcup_{a \in ee}\{\phi(a)\}$. Then, there exists $a \in ee$ such that $\phi(a) = \phi(e)$. It follows that $a = e$, because ϕ be a monomorphism. Hence, $e \in ee$ and therefore, e is an idempotent. ■

3. Some Results on Polygroups

In this section, the notion of polygroups are presented and some results of polygroup are established.

Definition 3.1: (Comer, 1996) Let P be a non-empty set. A *polygroup* is a system $\langle P, \bullet, e, ^{-1}\rangle$, where $e \in P$, $\bullet$ maps $P \times P$ into the non-empty subsets of P, i.e., $\emptyset \neq x \bullet y = \bullet(x, y) \subseteq P$ for all $x, y \in P$, and the following axioms hold for all x, y, z in P:

(i) $(x \bullet y) \bullet z = x \bullet (y \bullet z)$;

(ii) $e \bullet x = x \bullet e = \{x\}$;

(iii) for each $x \in P$ there exists a unique $x^{-1} \in P$ such that $e \in x \bullet x^{-1}$ and $e \in x^{-1} \bullet x$;

(iv) $x \in y \bullet z$ implies $y \in x \bullet z^{-1}$ and $z \in y^{-1} \bullet x$.

The following elementary facts about polygroups follow easily from the axioms: For all $, y \in P$, $e \in x \bullet x^{-1} \cap x^{-1} \bullet x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, $(x \bullet y)^{-1} = y^{-1} \bullet x^{-1}$ and $x \in y \bullet z$ implies $x^{-1} \in z^{-1} \bullet y^{-1}$.

Example 3.2: Let $P = \{a, b, c, d\}$ with the following multiplication table:

$\cdot$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	c	$\{a, b, d\}$	$\{c, d\}$
d	d	d	$\{c, d\}$	$\{a, b, c\}$

Then, P is a polygroup.

Definition 3.3: (Davvaz, 2000) A non-empty subset K of a polygroup P is said to be a *subpolygroup* of P if, under the hyperoperation in P, K itself forms a polygroup.

From now on, for x, y in a polygroup $\langle P, \bullet, e, ^{-1}\rangle$, we write the product of x, y as xy instead of $x \bullet y$.

Lemma 3.4: (Davvaz, 2000) A non-empty subset K of a polygroup P is a subpolygroup of P if and only if

(i) $a, b \in K$ implies $ab \subseteq K$;

(ii) $a \in K$ implies $a^{-1} \in K$.

Lemma 3.5: A non-empty subset K of a polygroup P is a subpolygroup of P if and only if $ab^{-1} \subseteq K$ for all $a, b \in K$.

Proof. Suppose that K is a subpolygroup of a polygroup P. Let, $\in K$. By Lemma 3.4 (ii), we have $b^{-1} \in K$, and thus, $ab^{-1} \subseteq K$. Conversely, assume that $ab^{-1} \subseteq K$ for all, $\in K$. Let $a, b \in K$. Then, $ab^{-1} \subseteq K$. Since $a \in K \subseteq P$, $e \in aa^{-1} \subseteq K$. This implies that $ea^{-1} \subseteq K$. It follows that $\{a^{-1}\} \subseteq K$, because $ea^{-1} = \{a^{-1}\}$. Thus, $a^{-1} \in K$. We also have that $b^{-1} \in K$. It obtains $ab = a(b^{-1})^{-1} \subseteq K$. Hence, by Lemma 3.4, K is a subpolygroup of P. ■

Lemma 3.6: Let S be a semigroup satisfies the following conditions:

(i) there exists $e \in S$ such that $xe = \{x\}$ for all $x \in S$;

(ii) for each $x \in S$ there exists $x^{-1} \in S$ such that $e \in xx^{-1}$;

(iii) for all $x, y, z \in S$, $x \in yz$ implies $y \in xz^{-1}$ and $z \in y^{-1}x$.

Then, S is a polygroup.

Proof. First, let $x, y \in S$, we will show that $e^{-1} = e$ and $(x^{-1})^{-1} = x$. From (i), we have $e \in ee$. Then, by (iii), $e \in e^{-1}e = \{e^{-1}\}$, thus, $e^{-1} = e$. Since $x \in xe$, $e \in x^{-1}x$. Again, by (iii), $x \in (x^{-1})^{-1}e = \{(x^{-1})^{-1}\}$ and we get that $x = (x^{-1})^{-1}$.

Suppose that $p \in S$ such that $e \in xp$ and $e \in px$. This implies that $p \in x^{-1}e = \{x^{-1}\}$, that is $p = x^{-1}$. Hence, x^{-1} is unique. Finally, let us show that $ex = \{x\}$. Since $e \in xx^{-1}$, $x \in e(x^{-1})^{-1} = ex$. Hence, $\{x\} \subseteq ex$. Let $a \in ex$. Then, $e \in ax^{-1}$ and $x^{-1} \in a^{-1}e = \{a^{-1}\}$. Certainly, we obtain that $x^{-1} = a^{-1}$. It is immediately that $e \in xa^{-1}$ and $e \in a^{-1}x$. Since $e \in a^{-1}x$, we have that $x \in (a^{-1})^{-1}e = ae$, and hence, $a \in xe^{-1} = xe = \{x\}$. Thus, $ex \subseteq \{x\}$ and then $ex = \{x\}$. Therefore, S is a polygroup. ■

Lemma 3.7: If $\langle S, \bullet, e, ^{-1} \rangle$ is a polygroup and ρ is an equivalence relation on S, then ρ is regular if and only if $\langle S/\rho, \otimes, \rho(e), ^{-I} \rangle$ is a polygroup, where $\rho(a)^{-I} = \rho(a^{-1})$.

Proof. Let $\langle S, \bullet, e, ^{-1} \rangle$ be a polygroup and ρ be an equivalence relation on S. Suppose that ρ is regular. We have $(S/\rho, \otimes)$ is a semipolygroup. Since $e \in S$, $\rho(e) \in S/\rho$. Then, we have that $\rho(x) \otimes \rho(e) = \{\rho(a) : a \in xe = \{x\}\} = \{\rho(x)\}$ and $\rho(e) \otimes \rho(x) = \{\rho(b) : b \in ex = \{x\}\} = \{\rho(x)\}$ for all $\rho(x) \in S/\rho$. Let $\rho(x) \in S/\rho$. Then, $x \in S$ and hence, there exists a unique $x^{-1} \in S$ such that $e \in xx^{-1}$ and $e \in x^{-1}x$. That is, there is $\rho(x)^{-I} = \rho(x^{-1}) \in S/\rho$ such that $\rho(e) \in \rho(x) \otimes \rho(x)^{-I} = \rho(x) \otimes \rho(x^{-1}) = \{\rho(a) : a \in xx^{-1}\}$ and $\rho(e) \in \rho(x)^{-I} \otimes \rho(x) = \rho(x^{-1}) \otimes \rho(x) = \{\rho(a) : a \in x^{-1}x\}$. Next, let $\rho(y) \in S/\rho$ be such that $\rho(e) \in \rho(x) \otimes \rho(y)$ and $\rho(e) \in \rho(y) \otimes \rho(x)$. Then, $e \in xy$ and $e \in yx$. It follows that $y = x^{-1}$ and $\rho(y) = \rho(x^{-1})$ is unique. Now, we let $\rho(x), \rho(y), \rho(z) \in S/\rho$ be such that $\rho(x) \in \rho(y) \otimes \rho(z)$. Then, $x \in yz$ and thus, $y \in xz^{-1}$ and $z \in y^{-1}x$. It follows that $\rho(y) \in \rho(x) \otimes \rho(z^{-1}) = \rho(x) \otimes \rho(z)^{-I}$ and $(z) \in \rho(y^{-1}) \otimes \rho(x) = \rho(y)^{-I} \otimes \rho(x)$. Hence, $\langle S/\rho, \otimes, \rho(e), ^{-I} \rangle$ is a polygroup.

Conversely, we assume that $\langle S/\rho, \otimes, \rho(e), ^{-I} \rangle$ is a polygroup, where $\rho(a)^{-I} = \rho(a^{-1})$. Let a, b and x be arbitrary elements of S such that $a\rho b$. If $u \in ax$, then $\rho(u) \in \rho(a) \otimes \rho(x) = \rho(b) \otimes \rho(x) = \{\rho(v): v \in bx\}$. Therefore, there exists $v \in bx$ such that $\rho(u) = \rho(v)$, i.e., $u\rho v$. If $v' \in bx$, then $\rho(v') \in \rho(b) \otimes \rho(x) = \rho(a) \otimes \rho(x) = \{\rho(u'): u' \in ax\}$. Therefore, there exists $u' \in ax$ such that $\rho(u') = \rho(v')$, i.e., $u'\rho v'$. Thus, $(ax)\bar{\rho}(bx)$ and hence, ρ is regular on the right. Similarly, we obtain that ρ is regular on the left. Hence, ρ is regular. ■

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5. References

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