A Remark on a Functional Equation Related to Digital Filtering

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Abstract

In this paper, by using the technique of Haruki and Nakagiri in 2007, we show that the functional equation related to digital filtering

$$f(x+t,y+t)+f(x-t,y)+f(x,y-t)=f(x-t,y-t)+f(x,y+t)+f(x+t,y)$$

is equivalent to the following equations:

$$f(x+2t,y+t)+f(x,y+t)+f(x-2t,y)+f(x+t,y-t)+f(x-t,y-t)$$
$$=f(x-2t,y-t)+f(x,y-t)+f(x+2t,y)+f(x-t,y+t)+f(x+t,y+t),$$

and

$$f(x+t,y+2t)+f(x-t,y+t)+f(x-t,y-t)+f(x+t,y)+f(x,y-2t)$$
$$=f(x-t,y-2t)+f(x+t,y-t)+f(x+t,y+t)+f(x-t,y)+f(x,y+2t).$$

Keywords: functional equation; translation (shift) operators; functional equation related to digital filtering; wave equation

1. Introduction

A functional equation is an equation in which the unknown (or unknowns) are functions. The best known functional equation is connected to the famous Cauchy functional equation, whose additive form, referred to as the additive Cauchy functional equation, is

$$f(x+y)=f(x)+f(y), \qquad (1.1)$$

and whose solution is collectively referred to as an additive function. In this paper, we will consider the geometric functional equation, which have been examined by many authors. First, in 1968, Aczél *et al*. studied the general solution of the functional equation

$$f(x+t,y+t)+f(x+t,y-t)+f(x-t,y+t)$$
$$+f(x-t,y-t)=4f(x,y), \qquad (1.2)$$

where $f:\mathbb{R}^2\to\mathbb{R}$ and x,y,t are real variables. The general solution of (1.2) is given in terms of arbitrary symmetric multi-additive functions of four variables. The few years later, Haruki (1970) (see also Kannappan, 2008), considered the wave equation,

$$f(x+t,y)+f(x-t,y)$$
$$=f(x,y+t)+f(x,y-t), \qquad (1.3)$$

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DOI: 10.14456/tjst.2013.2

where $f:\mathbb{R}^2 \longrightarrow \mathbb{R}$, and the general solution of (1.3) is given by

$$f(x,y)=\alpha(x,y)+\beta(x-y)+B(x,y)$$

where $\alpha,\beta:\mathbb{R}\longrightarrow\mathbb{R}$ are arbitrary functions and $B:\mathbb{R}^2 \longrightarrow \mathbb{R}$ is biadditive and skew-symmetric.

It is fascinating to known that functional equations arise in many fields of applied science, such as physics, statistics, economic, engineering and computer. Especially, the computer vision is a relatively new and fast growing field which primarily aids computers with artificial sensory perception. In other words, the computer vision deals with image understanding. Preprocessing is a basic task in every image understanding problem. Among others, preprocessing involves filtering. Spatial filters are used for deblurring, smoothing, sharpening and enhancing of images. Thus filtering is a technique for modifying or enhancing an image.

In 2001, Sahoo and Székelyhidi, determined the general complex-valued solution of the functional equation related to digital filtering,

$$f(x+t,y+t)+f(x-t,y)+f(x,y-t)$$
$$=f(x-t,y-t)+f(x,y+t)+f(x+t,y), \quad \text{(W1)}$$

for all $x,y,t\in G$ and $f:G\times G\longrightarrow\mathbb{C}$ where G is a 2-divisible abelian group and $\mathbb{C}$ the set of complex numbers. The general solution is given by $f(x,y)=B(x,y)+\phi(x)+\psi(y)+\chi(x-y)$, for all $x,y\in G$, where $B:G\times G\longrightarrow\mathbb{C}$ is biaddtive and $\phi,\psi,\chi:G\longrightarrow\mathbb{C}$ are arbitrary functions. Next, in 2007, Haruki and Nakagiri considered the pexider type generalization of the wave equation (1.3), that is the following equation,

$$f_1(x+t,y)+f_2(x-t,y)$$
$$=f_3(x,y+t)+f_4(x,y-t), \quad (1.4)$$

for all $x,y,t\in G$ and $f_1,f_2,f_3,f_4:G\times G\longrightarrow\mathbb{C}$ where G is a 2-divisible abelian group and $\mathbb{C}$ the set of complex numbers. Their result is:

Theorem 1.1 If $f_1,f_2,f_3,f_4:G\times G\longrightarrow\mathbb{C}$ satisfy (1.3) for all $x,y,t\in G$, then there exist

(i) a skew-symmetric biadditive function $A:G\times G\longrightarrow\mathbb{C}$,

(ii) biadditive functions $B_1,B_2:G\times G\longrightarrow\mathbb{C}$, and

(iii) arbitrary functions $\alpha,\beta,\phi_1,\phi_2,\psi_1,\psi_2,\chi_1,\chi_2:G\longrightarrow\mathbb{C}$ such that

$$\left.\begin{aligned}
f_1(x,y)&=A(x,y)+B_1(x+y,x-y)\\
&\quad+\alpha(x+y)+\beta(x-y)+\phi_1(x+y)\\
&\quad+\psi_1(x-y)+\chi_1(2y),\\
f_2(x,y)&=A(x,y)-B_1(x+y,x-y)\\
&\quad+\alpha(x+y)+\beta(x-y)-\phi_1(x+y)\\
&\quad-\psi_1(x-y)-\chi_1(2y),\\
f_3(x,y)&=A(x,y)+B_2(x+y,2x)\\
&\quad+\alpha(x+y)+\beta(x-y)+\phi_2(x+y)\\
&\quad+\psi_2(2x)+\chi_2(-x+y),\\
f_4(x,y)&=A(x,y)-B_2(x+y,2x)\\
&\quad+\alpha(x+y)+\beta(x-y)-\phi_2(x+y)\\
&\quad-\psi_2(2x)-\chi_2(-x+y).
\end{aligned}\right\} \quad (1.5)$$

However, in Haruki and Nakagiri's paper, they consider (1.3) under the assumption that $f:G\times G\longrightarrow\mathbb{C}$, and they proved that the equation (1.3) is equivalent to each one of the following two equations:

$$f(x+t,y+t)+f(x+t,y-t)$$

$$+f(x-t,y+t)+f(x-t,y-t) \quad (1.6)$$
$$=f(x,y+2t)+f(x,y-2t)+2f(x,y)$$

and

$$f(x+t,y+t)+f(x+t,y-t)$$
$$+f(x-t,y+t)+f(x-t,y-t) \quad (1.7)$$
$$=f(x+2t,y)+f(x-2t,y)+2f(x,y)$$

Our aim in this paper, by using the same technique of Haruki and Nakagiri in 2007, is to show that the functional equation (W1) is equivalent to each one of the following two equations:

$$f(x+2t,y+t)+f(x,y+t)+f(x-2t,y)$$
$$+f(x+t,y-t)+f(x-t,y-t) \quad \text{(W2)}$$
$$=f(x-2t,y-t)+f(x,y-t)+f(x+2t,y)$$
$$+f(x-t,y+t)+f(x+t,y+t),$$

and

$$f(x+t,y+2t)+f(x-t,y+t)+f(x-t,y-t)$$
$$+f(x+t,y)+f(x,y-2t) \quad \text{(W3)}$$
$$=f(x-t,y-2t)+f(x+t,y-t)$$
$$+f(x+t,y+t)+f(x-t,y)+f(x,y+2t).$$

2. Equivalence of equations (W1), (W2) and (W3)

Throughout this paper $(G,+)$ will denote a 2-divisible abelian group and $\mathbb{C}$ the field of all complex numbers. It is convenient to introduce translation (shift) operators X^t and Y^t for $t\in G$ defined by $X^t f(x,y)=f(x+t,y)$ and $Y^t f(x,y)=f(x,y+t)$ for all $x,y\in G$. In particular $1=X^0=Y^0$ denote the identity operators. Our main result is:

Theorem 2.1 Equations (W1), (W2) and (W3) for all $x,y,t\in G$ are equivalent to each other under the assumption $f:G\times G\rightarrow\mathbb{C}$.

Proof. First, we will show that (W1) is equivalent to (W2). We can write (W1) in the following operator form

$$(X^tY^t+X^{-t}+Y^{-t})f(x,y)$$
$$=(X^{-t}Y^{-t}+X^t+Y^t)f(x,y). \quad (2.1)$$

Multiplying (2.1) by X^t, we obtain that

$$(X^{2t}Y^t+1+X^tY^{-t})$$
$$=(Y^{-t}+X^{2t}+X^tY^t)f(x,y). \quad (2.2)$$

Again, multiplying (2.1) by X^{-t}, we get

$$(Y^t+X^{-2t}+X^{-t}Y^{-t})f(x,y)$$
$$=(X^{-2t}Y^{-t}+1+X^{-t}Y^t)f(x,y). \quad (2.3)$$

Then, adding (2.2) and (2.3), we get

$$(X^{2t}Y^t+Y^t+X^{-2t}+X^tY^{-t}+X^{-t}Y^{-t})f(x,y)$$
$$=(X^{-2t}Y^{-t}+Y^{-t}+X^{2t}+X^{-t}Y^t+X^tY^t)f(x,y), \quad (2.4)$$

which is the operator form of (W2). Thus (W1) implies (W2).

Conversely, squaring both sides of (2.4), we have

$$[(X^{2t}Y^t+Y^t+X^{-2t}+X^tY^{-t}+X^{-t}Y^{-t})$$
$$(X^{2t}Y^t+Y^t+X^{-2t}+X^tY^{-t}+X^{-t}Y^{-t})]f(x,y)$$
$$=[(X^{-2t}Y^{-t}+Y^{-t}+X^{2t}+X^tY^t+X^{-t}Y^t)$$
$$(X^{-2t}Y^{-t}+Y^{-t}+X^{2t}+X^{-t}Y^t+X^tY^t)]f(x,y).$$

Then, we get

$$[X^{4t}Y^{2t}+2X^{2t}Y^{2t}+2Y^t+2X^{3t}+4X^t$$
$$+Y^{2t}+2X^{-2t}Y^t+2X^{-t}+X^{-4t}$$
$$+2X^{-t}Y^{-t}+2X^{-3t}Y^{-t}+X^{2t}Y^{-2t}$$
$$+2Y^{-2t}+X^{-2t}Y^{-2t}]f(x,y) \quad (2.5)$$
$$=[X^{-4t}Y^{-2t}+2X^{-2t}Y^{-2t}+2Y^{-t}$$
$$+2X^{-3t}+4X^{-t}+Y^{-2t}+2X^{2t}Y^{-t}$$
$$+2X^t+X^{4t}+2X^tY^t+2X^{3t}Y^t+X^{-2t}Y^{2t}$$
$$+2Y^{2t}+X^{2t}Y^{2t}]f(x,y).$$

Replacing t by $2t$ in (2.4), which is the operator form of (W2), we obtain that

$$(X^{4t}Y^{2t}+Y^{2t}+X^{-4t}+X^{2t}Y^{-2t}+X^{-2t}Y^{-2t})f(x,y) \quad (2.6)$$
$$=(X^{-4t}Y^{-2t}+Y^{-2t}+X^{4t}+X^{-2t}Y^{2t}+X^{2t}Y^{2t})f(x,y).$$

If we subtract (2.6) from (2.5), then

$$[X^{2t}Y^{2t}+Y^{t}+X^{3t}+X^{t}+X^{-2t}Y^{t}+X^{-t}Y^{-t}+X^{-3t}Y^{-t}+Y^{-2t}]f(x,y) \quad (2.7)$$
$$=[X^{-2t}Y^{-2t}+Y^{-t}+X^{-3t}+X^{-t}+X^{2t}Y^{-t}+X^{t}Y^{t}+X^{3t}Y^{t}+Y^{2t}]f(x,y).$$

Again, multiplying (W2) by $X^{t}+X^{-t}$, we get

$$(X^{3t}Y^{t}+X^{t}Y^{t}+X^{-t}+X^{2t}Y^{-t}+Y^{-t}+X^{t}Y^{t}+X^{-t}Y^{t}+X^{-3t}+X^{-2t}Y^{-t}+Y^{-t})f(x,y) \quad (2.8)$$
$$=(X^{-t}Y^{-t}+X^{t}Y^{-t}+X^{3t}+Y^{t}+X^{2t}Y^{t}+X^{-3t}Y^{-t}+X^{-t}Y^{-t}+X^{t}+X^{-2t}Y^{t}+Y^{t})f(x,y).$$

Then, adding (2.7) and (2.8), we obtain that

$$[X^{2t}Y^{2t}+Y^{-2t}+X^{-t}Y^{t}+X^{t}Y^{t}+Y^{-t}+X^{-2t}Y^{-t}]f(x,y) \quad (2.9)$$
$$=[X^{-2t}Y^{-2t}+Y^{2t}+X^{t}Y^{-t}+Y^{t}+X^{-t}Y^{-t}+X^{2t}Y^{t}]f(x,y).$$

Now, adding (2.9) and (W2), we get

$$[X^{2t}Y^{2t}+Y^{-2t}+X^{-2t}]f(x,y)$$
$$=[X^{-2t}Y^{-2t}+Y^{2t}+X^{2t}]f(x,y). \quad (2.10)$$

Replacing $2t$ by t, so (2.10) yields

$$[X^{t}Y^{t}+Y^{-t}+X^{-t}]f(x,y)$$
$$=[X^{-t}Y^{-t}+Y^{t}+X^{t}]f(x,y), \quad (2.11)$$

which is (W1). Hence, (W1) is equivalent to (W2).

Now, we will show that (W1) is equivalent to (W3), as before, multiplying (2.1) by Y^{t}, we obtain that

$$(X^{t}Y^{2t}+X^{-t}Y^{t}+1)f(x,y)$$
$$=(X^{-t}+X^{t}Y^{t}+Y^{2t})f(x,y). \quad (2.12)$$

Multiplying (2.1) by Y^{-t}, we obtain that

$$(X^{t}+X^{-t}Y^{-t}+Y^{-2t})f(x,y)$$
$$=(X^{-t}Y^{-2t}+X^{t}Y^{-t}+1)f(x,y). \quad (2.13)$$

Adding (2.12) and (2.13), we get

$$(X^{t}Y^{2t}+X^{-t}Y^{t}+X^{-t}Y^{-t}+X^{t}+Y^{-2t})f(x,y)$$
$$=(X^{-t}Y^{-2t}+X^{t}Y^{-t}+X^{t}Y^{t}+X^{-t}+Y^{2t})f(x,y), \quad (2.14)$$

which is the operator form of (W3), that is, (W1) implies (W3).

On the other hand, squaring both sides of (2.14), we get

$$[X^{2t}Y^{4t}+2Y^{3t}+4Y^{t}+2X^{2t}Y^{2t}+2X^{t}+X^{-2t}Y^{2t}+2X^{-2t}+2X^{-t}Y^{-t}+X^{-2t}Y^{-2t}+2Y^{-t}+2X^{-t}Y^{-3t}+X^{2t}+2X^{t}Y^{-2t}+Y^{-4t}]f(x,y)$$
$$=[X^{-2t}Y^{-4t}+2Y^{-3t}+4Y^{-t}+2X^{-2t}Y^{-2t}+2X^{-t}+X^{2t}Y^{-2t}+2X^{2t}+2X^{t}Y^{t}+X^{2t}Y^{2t}+2Y^{t}+2X^{t}Y^{3t} \quad (2.15)$$

Replacing t by $2t$ in (2.14), which is the operator form of (W3), we have

$$(X^{2t}Y^{4t}+X^{-2t}Y^{2t}+X^{-2t}Y^{-2t}+X^{2t}+Y^{-4t})f(x,y)$$
$$=(X^{-2t}Y^{-4t}+X^{2t}Y^{-2t}+X^{2t}Y^{2t}+X^{-2t}+Y^{4t})f(x,y). \quad (2.16)$$

If we subtract (2.16) from (2.15), then

$$[Y^{3t}+Y^{t}+X^{2t}Y^{2t}+X^{t}+X^{-2t}+X^{-t}Y^{-t}+X^{-t}Y^{-3t}+X^{t}Y^{-2t}]f(x,y) \quad (2.17)$$
$$=[Y^{-3t}+Y^{-t}+X^{-2t}Y^{-2t}+X^{-t}+X^{2t}+X^{t}Y^{t}+X^{t}Y^{3t}+X^{-t}Y^{2t}]f(x,y).$$

Next, multiplying (W3) by $Y^t + Y^{-t}$, we get

$$(X^tY^{3t} + X^{-t}Y^{2t} + X^{-t} + X^tY^t + Y^{-t} + X^tY^t + X^{-t} + X^{-t}Y^{-2t} + X^tY^{-t} + Y^{-3t})f(x,y)$$
$$=(X^{-t}Y^{-t} + X^t + X^tY^{2t} + X^{-t}Y^t + Y^{3t} + X^{-t}Y^{-3t} + X^tY^{-2t} + X^t + X^{-t}Y^{-t} + Y^t)f(x,y). \quad (2.18)$$

Adding (2.17) and (2.18), we get

$$[X^{2t}Y^{2t} + X^{-2t} + X^tY^t + X^{-t} + X^{-t}Y^{-2t} + X^tY^{-t}]f(x,y) \quad (2.19)$$
$$=[X^{-2t}Y^{-2t} + X^{2t} + X^{-t}Y^{-t} + X^t + X^tY^{2t} + X^{-t}Y^t]f(x,y).$$

Again, adding (2.19) and (W3), we get

$$[X^{2t}Y^{2t} + X^{-2t} + Y^{-2t}]f(x,y) = [X^{-2t}Y^{-2t} + X^{2t} + Y^{2t}]f(x,y). \quad (2.20)$$

Finally, replacing $2t$ by t, then (2.20) yields

$$[X^tY^t + X^{-t} + Y^{-t}]f(x,y) = [X^{-t}Y^{-t} + X^t + Y^t]f(x,y). \quad (2.21)$$

The above equation implies (2.1), which is (W1). Hence, (W1) is equivalent to (W3). This completes the proof.

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3. Conclusion

Using the technique of Haruki and Nakagiri in 2007, the equivalence of three functional equations (W1), (W2) and (W3) are investigated.

4. References

Aczèl, J., Haruki, H., McKiernan, M.A. and Sakovic, G.N., 1968, General and regular solutions of function equations characterizing harmonic polynomials, Aequationes Math. 1: 37-53.

Haruki, S. 1970, On the functional equation $f(x+t,y)+f(x-t,y)=f(x,y+t)+f(x,y-t)$, Aequationes Math. 5: 118-119.

Haruki, S. and Nakagiri, S. 2007, A pexiderized wavelike partial difference functional equation, Aequationes Math. 74: 1-6.

Kannappan, P., 2008, Functional Equations and Inequalities with Applications, Springer, New York.

Sahoo, P.K. and Szèkelyhidi, L., 2001, On a functional equation related to digital filtering, Aequationes Math. 62: 280-285.