A General Solution of a Generalized Mixed Type of Polynomial Functional Equations

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Abstract

In this paper, we determine the general solution of the following functional equation $f_1(x+3y)+f_1(x-3y)+f_2(x+2y)+f_2(x-2y)+f_3(x)=f_4(x+y)+f_4(x-y)+f_5(y)$ without assuming any regularity condition on the unknown functions $f_1,f_2,f_3,f_4,f_5 : X \to Y$, where X and Y are real vector spaces.

Keywords: functional equation; Fréchet functional equation; quartic functional equation; additive function; polynomial functional equations

1. Introduction

For a function $f:\mathbb{R}\to\mathbb{R}$, the difference operator Δ_h with $h\in\mathbb{R}$ is defined by $\Delta_h f(x)=f(x+h)-f(x)$. Furthermore, let $\Delta_h^0 f(x)=f(x), \Delta_h^1 f(x)=\Delta_h f(x)$ and $\Delta_h \circ \Delta_h^n f(x)=\Delta_h^{n+1} f(x)$ for all $n\in\mathbb{N}$ and for all $h\in\mathbb{R}$. The superposition of difference operations is defined by $\Delta_{h_1,\ldots,h_n} f=\Delta_{h_1}\circ\Delta_{h_2}\circ\ldots\circ\Delta_{h_n} f, (n\in\mathbb{N})$, where $\Delta_{h_i}\circ\Delta_{h_j}$ denotes the composition. If $h_1=h_2=\ldots=h_n=h$, we write $\Delta_{\underbrace{h,\ldots,h}_{n}} f=\Delta_h^n f$.

For any given $n\in\mathbb{N}\cup\{0\}$, if f satisfies the functional equation

(1.1) $\Delta_h^{n+1} f(x)=0 \qquad (x,h\in\mathbb{R}),$

Then f is called a *polynomial function* of order n. In explicit form (1.1) can be written as

(1.2) $\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k} f(x+kh)=0.$

It is known (see [8]) that with functions defined over $\mathbb{R}$, the equation (1.2) is equivalent to the Fréchet functional equation

(1.3) $\Delta_{h_1,\ldots,h_{n+1}} f(x)=0 \qquad (x,h_1,\ldots,h_{n+1}\in\mathbb{R}).$

We quote three results from [3, pp.71-77, Theorems 9.3, 9.4, 9.6] which are needed in our work here.

Theorem 1.1 Let X and Y be two linear spaces. Then

I. If $f:X\to Y$ is a polynomial function of order n, then $\Delta_{h_1,\ldots,h_{n+1}} f(x)=0$ for $x,h_1,\ldots,h_{n+1}\in X.$

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II. If $A_k : X^k \to Y\ (k=0,1,...,n)$ are symmetric k-additive functions and if A^k are their diagonalizations, then the function $\sum_{k=0}^{n} A^k(x)$ is a polynomial function of order n.

III. If $f : X \to Y$ is a polynomial function of order n, then there exist k-additive symmetric functions $A_k : X^k \to Y\ (k=0,1,...,n)$ such that $f(x) = \sum_{k=0}^{n} A^k(x)$, where A^k are the diagonalizations of A_k.

The following proposition give properties of an n-additive function and its diagonalization [for detail, please refer to Czerwik (2001)].

Proposition 1.2 Let $A^n : \mathbb{R} \to \mathbb{R}$ be a diagonalization of n-additive function A_n. Then

I. For all $x \in \mathbb{R}$ and for all $r \in \mathbb{Q}$,
$A^n(rx) = r^n A^n(x)$.

II. For all $x, y \in \mathbb{R}$ and for all $r, s \in \mathbb{Q}$,

$$A^n(rx+sy) = \sum_{i=0}^{n} r^{n-i} s^i \binom{n}{i} A^{n-i,i}(x,y).$$

Furthermore the resulting function after substitution $x_1 = x_2 = ... = x_l = x$ and $x_{l+1} = x_{l+2} = ... = x_n = y$ in $A_n(x_1, x_2, ..., x_n)$ will be denoted by $A^{l,n-l}(x,y)$.

Our work here is originated from the best known functional equation are those related to identity satisfied by a quadratic function and commonly called a quadratic functional equation,

(1.4) $q(x+y) + q(x-y) = 2q(x) + 2q(y)$.

In particular, every solution of the quadratic functional equation is said to be a quadratic function (Kannappan, 2008). There have been numerous related works. Let us mention some which are of interest to us. In 2005, Sahoo considered the functional equation

(1.5) $f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x)\ (x,y\in\mathbb{R})$.

It is easy to see that the function $f(x) = x^3$ is a solution of (1.5). Then (1.5) is called a cubic functional equation and every solution of (1.5) is called a cubic function. Sahoo proved that the function $f : \mathbb{R} \to \mathbb{R}$ satisfies (1.5) for all $x, y \in \mathbb{R}$ if and only if f is of the form $f(x) = A^3(x), \forall x \in \mathbb{R}$, where $A^3(x)$ is the diagonal of 3-additive symmetric map $A_3 : \mathbb{R}^3 \to \mathbb{R}$. Next, in 2008, Wiwatwanich and Nakmahachalasint established the general solution of the following cubic functional equation

(1.6) $f(x+3y)-3f(x+y)+3f(x-y)-f(x-3)=48f(y)$.

In 2001, Rassias introduced the new cubic mappings $f : X \to Y$, satisfying the cubic functional equation

(1.7) $f(x+2y)-3f(x+y)+3f(x)-f(x-y)=6f(y)\ (x,y\in X)$,

where X is a linear space and Y be a real complete linear space.

If we let $f(x) = x^4$, then the identity $(x+2y)^4+(x-2y)^4+6x^4 = 4\left[(x+y)^4+(x-y)^4+6y^4\right]$, yields a functional equation

(1.8) $f(x+2y)+f(x-2y)+6f(x)=4[f(x+y)+f(x-y)+6f(y)]$

for all $x, y \in \mathbb{R}$. For obvious reasons, the equation (1.8) is called a quartic functional equation. In 1999, Rassias introduced the quartic functional equation (1.8) and obtained a solution of the Hyers-Ulam stability problem for this quartic functional equation. Then, in 2003, Chung and Sahoo determined the quartic functional equation

$f(x+2y)+f(x-2y)+6f(x)=4\left[f(x+y)+f(x-y)+6f(y)\right]\ (x,y\in\mathbb{R})$

without assuming any regularity conditions on the unknown function f ; their argument covers the case in which $f:G\to S$, where G and S are uniquely divisible abelian groups.

In 2004, Sahoo considered the functional equation characterizing polynomials of degree 3

(1.9) $f(x+2y)+f(x-2y)+6f(x)=4[f(x+y)+f(x-y)]\ (x,y\in\mathbb{R})$,

and proved that its general solution is of the form $f(x)=A^0+A^1(x)+A^2(x)+A^3(x)$, where $A^n(x)$ is the diagonal of the n-additive symmetric function $A_n:\mathbb{R}^n\to\mathbb{R}\ (n=1,2,3)$ and A^0 is an arbitrary constant. One year later, he generalized (1.5) to

(1.10) $f_1(2x+y)+f_2(2x-y)=f_3(x+y)+f_4(x-y)+f_5(x)\quad (x,y\in\mathbb{R})$,

and proved that $f_1,f_2,f_3,f_4,f_5:\mathbb{R}\to\mathbb{R}$ satisfy (1.10) if and only if

$f_1(x)=A^3(x)+A^2(x)+A^1(x)+A^0+B^2(x)+B^1(x)+B^0$,

$f_2(x)=A^3(x)+A^2(x)+A^1(x)+A^0-B^2(x)-B^1(x)-B^0$,

$f_3(x)=2A^3(x)+A^2(x)+A^1(x)+\frac{1}{2}A^0+C^1(x)+C^0+2B^2(x)+B^1(x)+B^0+D^0$,

$f_4(x)=2A^3(x)+A^2(x)+A^1(x)+\frac{1}{2}A^0+C^1(x)+C^0-2B^2(x)-B^1(x)-B^0-D^0$,

$f_5(x)=12A^3(x)+6A^2(x)+2A^1(x)+A^0-2C^1(x)-2C^0$,

where A^0,B^0,C^0,D^0 are arbitrary constants, $A^n(x),B^n(x),C^n(x)$ are the diagonals of n-additive symmetric functions A_n,B_n,C_n, respectively, $n=1,2,3$.

In the few years later, Xu, Rassias, and Xu (2011) determined the general solution of the functional equation,

(1.11) $f(x+ky)+f(x-ky)=g(x+y)+g(x-y)+h(x)+\tilde{h}(y)$

for fixed integers k with $k\neq 0,\pm 1$. Their results are:

Theorem 1.3. Let X and Y be a real vector spaces. If functions $f,g,h,\tilde{h}:X\to Y$ satisfy (1.11) for all $x,y\in X$ for fixed integers k such that $k\neq 0,\pm 1$, then f is a solution of the Fréchet functional equation $\Delta_{x_1,x_2,x_3,x_4,x_5}f(x_0)=0$ for all $x_0,x_1,x_2,x_3,x_4,x_5\in X$.

Theorem 1.4. Let X and Y be a real vector spaces. Function $f,g,h,\tilde{h}:X\to Y$ satisfy (1.11) for all $x,y\in X$ for fixed integers k such that $k\neq 0,\pm 1$ if and only if

$f(x)=A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0$,

$g(x)=k^2A^4(x)+k^2A^3(x)+B^2(x)+B^0+C^1(x)+D^0$,

$h(x)=(2-2k^2)A^4(x)+(2-2k^2)A^3(x)+$
$+2A^2(x)+2A^1(x)+2A^0-2B^2(x)-2C^1(x)-2B(x)$,

$\tilde{h}(x)=(2k^4-2k^2)A^4(x)+2k^2A^2(x)-2B^2(x)-2D^0$,

where A^0,B^0,C^0,D^0 are arbitrary constants, $A^n(x),B^n(x),C^n(x)$ are the diagonals of n-additive symmetric functions A_n,B_n,C_n, respectively, $n=1,2,3,4$.

Moreover, they proved the useful following theorem:

Theorem 1.5. Let G and S be commutative groups, let n be a nonnegative integer, and let ϕ_i and ψ_i be additive functions from G into G such that $\phi_i(G)\subseteq\psi_i(G), i=1,2,,...,n+1.$ If functions $f,f_i:G\to S, i=1,2,...,n+1,$ satisfy the condition

$f(x)+\sum_{i=1}^{n+1} f_i\big(\phi_i(x)+\psi_i(y)\big)=0,$ then f satisfies the Fréchet functional equation $\Delta_{x_1,x_2,\ldots,x_{n+1}} f(x_0)=0.$

Recently, Hengkrawit and Thayacharoen (2012) considered the following quartic functional equation,

(1.12) $f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)=13\big(f(x+y)+f(x-y)\big)+168f(y)$

for all $x,y\in\mathbb{R}$, and proved that its general solution is of the form $f(x)=A^4(x)$, where $A^4(x)$ is the diagonal of 4-additive symmetric function $A_4:\mathbb{R}^4\to\mathbb{R}$.

This quartic functional equation (1.12) can be generalized to the following functional equation

(1.13) $f_1(x+3y)+f_1(x-3y)+f_2(x+2y)+f_2(x-2y)+f_3(x)=f_4(x+y)+f_4(x-y)+f_5(y)$

for all $x,y\in X$ and $f_1,f_2,f_3,f_4,f_5:X\to Y$ are unknown functions, where X and Y are real vector spaces.

Our aim is to find the general solution of the equation (1.13) and some other related functional equations. The method used for solving these functional equations is elementary but without using any regularity condition on the unknown functions.

2. Solution of equation (1.13)

In this section, we determine the general solution of the functional equation (1.13) and some other related equations without assuming any regularity conditions for the unknown functions. The following lemma holds from Theorem 1.5 and Theorem 1.1.

Lemma 2.1 Let X and Y be real vector spaces. If functions $f_1,f_2,\ldots,f_5:X\to Y$ satisfy the functional equation

(2.1) $f_1(x)+f_1(x+6y)+f_2(x+5y)+f_2(x+y)+f_3(x+3y)-f_4(x+4y)-f_4(x+2y)-f_5(y)=0$

for all $x,y\in X$ if and only if f_1 is of the form $f_1(x)=A^6(x)+A^5(x)+A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0,$ for all $x\in X$, where $A^i(x)$ is the diagonal of i-additive symmetric map $A_i:X^i\to Y$ for $i=1,2,\ldots,6.$

Proof. Assume that f_1 satisfies the functional equation (2.1).

Taking

$F_2(x+5y)=f_2(x+5y),$
$F_3(x+y)=f_2(x+y),\ F_4(x+3y)=f_3(x+3y),$
$F_5(x+4y)=-f_4(x+4y),\ F_6(x+2y)=-f_4(x+2y),$
$F_7(x)=-f_5(x),\ \phi_1=\phi_2=\phi_3=\phi_4=\phi_5=\phi_6=x, \phi_7=0,$
$\psi_1=6y,\psi_2=5y,\psi_3=y,\psi_4=3y,\psi_5=4y,\psi_6=2y,\psi_7=y.$

Where $F_1,F_2,\ldots,F_7:X\to Y$. We get

$F_1(x)+F_1(\phi_1(x)+\psi_1(y))+F_2(\phi_2(x)+\psi_2(y))+F_3(\phi_3(x)+\psi_3(y))+F_4(\phi_4(x)+\psi_4(y))+F_5(\phi_5(x)+\psi_5(y))+F_6(\phi_6(x)+\psi_6(y))+F_7(\phi_7(x)+\psi_7(y))=0$

Thus, we can rewrite the functional equation (2.1) in form $f_1(x)+\sum_{i=1}^{7} F_i(\phi_i(x)+\psi_i(y))=0$

From ϕ_i and ψ_i we see that $\phi_i(X)\subseteq\psi_i(X)$ for $i=1,2,\ldots,7$. By Theorem 1.5, f_1 satisfies the Fréchet functional equation $\Delta_{x_1,\ldots,x_7} f_1(x_0)=0$ for all $x_1,\ldots,x_7\in X$

Thus from Theorem 1.1, we have $f_1(x)=A^6(x)+A^5(x)+A^4(x)+A^3(x)+A^2(x)+A^1(x)+A^0,$ for all $x\in X$, where $A^i(x)$ is the diagonal of

$i-$additive symmetric map $A_i: X^i \to Y$ for $i=1,2,...,6.$ ❐

We are now ready to prove the main result.

Theorem 2.2 Let X and Y be real vector spaces. The functions $f_1, f_2, ..., f_5 : X \to Y$ satisfy the functional equation (1.13) for all $x, y \in X$ if and only if

$$f_1(x) = A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0,$$
$$f_2(x) = -6A^6(x) - 6A^5(x) + \left(\mathfrak{A}^4(x) - 6A^4(x)\right) + \left(\mathfrak{A}^3(x) - 6A^3(x)\right) + \left(\mathfrak{A}^2(x) - 6A^2(x)\right) + \mathfrak{A}^1(x) + \mathfrak{A}^0,$$
$$f_3(x) = -20A^6(x) - 20A^5(x) + \left(6\mathfrak{A}^4(x) - 20A^4(x)\right) + \left(6\mathfrak{A}^3(x) - 20A^3(x)\right) - \left(2\mathfrak{A}^2(x) + 20A^2(x)\right) - 2\left(\mathfrak{A}^1(x) + A^1(x)\right) - 2\left(\mathfrak{A}^0 + A^0\right) + 2B^2(x) + 2C^1(x) + 2B^0,$$
$$f_4(x) = -15A^6(x) - 15A^5(x) + \left(4\mathfrak{A}^4(x) - 15A^4(x)\right) + \left(4\mathfrak{A}^3(x) - 15A^3(x)\right) - 15A^2(x) + B^2(x) + B^0 + C^1(x) + D^0,$$
$$f_5(x) = 720A^6(x) + 24\mathfrak{A}^4(x) + 8\mathfrak{A}^2(x) - 2B^2(x) - 2D^0,$$

where $\mathfrak{A}^0, A^0, B^0, C^0, D^0$ are arbitrary constants, $\mathfrak{A}^n(x), A^n(x), B^n(x), C^n(x)$ are the diagonals of n-additive symmetric functions $\mathfrak{A}_n, A_n, B_n, C_n$, respectively, $n = 1, 2, ..., 6$.

Proof. Replacing x by $x+3y$ in (1.13), we obtain that $f_1(x) + f_1(x+6y) + f_2(x+5y) + f_2(x+y) + f_3(x+3y) - f_4(x+4y) - f_4(x+2y) - f_5(y) = 0$. Using Lemma 2.1, we have

$$f_1(x) = A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0,$$

For all $x \in X$, where $A^i(x)$ is the diagonal of i-additive symmetric map $A_i : X^i \to Y$ for $i=1,2,...,6.$

Consider $f_1(x+3y) + f_1(x-3y)$ and note that $A^6(x+3y) + A^6(x-3y) = 2A^6(x) + 270A^{4,2}(x,y) + 2430A^{2,4}(x,y) + 1458A^6(y)$, $A^5(x+3y) + A^5(x-3y) = 2A^5(x) + 180A^{3,2}(x,y) + 810A^{1,4}(x,y)$, $A^4(x+3y) + A^4(x-3y) = 2A^4(x) + 108A^{2,2}(x,y) + 162A^4(y)$, $A^3(x+3y) + A^3(x-3y) = 2A^3(x) + 54A^{1,2}(x,y)$, $A^2(x+3y) + A^2(x-3y) = 2A^2(x) + 18A^2(y)$. Then by the equation (1.13), we conclude that

(2.2) $f_4(x+y) + f_4(x-y) + f_5(y) - f_2(x+2y) - f_2(x-2y) - f_3(x) = 2A^6(x) + 270A^{4,2}(x,y) + 2430A^{2,4}(x,y) + 1458A^6(y) + 2A^5(x) + 180A^{3,2}(x,y) + 810A^{1,4}(x,y) + 2A^4(x) + 108A^{2,2}(x,y) + 162A^4(y) + 2A^3(x) + 54A^{1,2}(x,y) + 2A^2(x) + 18A^2(y) + 2A^1(x) + 2A^0.$
$= -15A^6(x+y) - 15A^6(x-y) + 6A^6(x+2y) + 6A^6(x-2y) + 20A^6(x) + 720A^6(y) - 15A^5(x+y) - 15A^5(x-y) + 6A^5(x+2y) + 6A^5(x-2y) + 20A^5(x) - 15A^4(x+y) - 15A^4(x-y) + 6A^4(x+2y) + 6A^4(x-2y) + 20A^4(x) - 15A^3(x+y) - 15A^3(x-y) + 6A^3(x+2y) + 6A^3(x-2y) + 20A^3(x) - 15A^2(x+y) - 15A^2(x-y) + 6A^2(x+2y) + 6A^2(x-2y) + 20A^2(x) + 2A^1(x) + 2A^0$, for all $x, y \in X$.

Define $F_2(x) = f_2(x) + 6\left(A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x)\right)$, $F_3(x) = -f_3(x) - 20\left(A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x)\right) - 2A^1(x) - 2A^0$, $F_4(x) = f_4(x) + 15\left(A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x)\right)$, $F_5(x) = f_5 - 720A^6(x).$

It follows from (2.2) that

(2.3) $F_2(x+2y) + F_2(x-2y) = F_4(x+y) + F_4(x-y) + F_3(x) + F_5(y).$

From Theorem 1.4, thus

$$f_1(x) = A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0,$$
$$f_2(x) = -6A^6(x) - 6A^5(x) + \left(\mathfrak{A}^4(x) - 6A^4(x)\right) + \left(\mathfrak{A}^3(x) - 6A^3(x)\right) + \left(\mathfrak{A}^2(x) - 6A^2(x)\right) + \mathfrak{A}^1(x) + \mathfrak{A}^0,$$
$$f_3(x) = -20A^6(x) - 20A^5(x) + \left(6\mathfrak{A}^4(x) - 20A^4(x)\right) + \left(6\mathfrak{A}^3(x) - 20A^3(x)\right) - \left(2\mathfrak{A}^2(x) + 20A^2(x)\right) - 2\left(\mathfrak{A}^1(x) + A^1(x)\right) - 2\left(\mathfrak{A}^0 + A^0\right) + 2B^2(x) + 2C^1(x) + 2B^0,$$
$$f_4(x) = -15A^6(x) - 15A^5(x) + \left(4\mathfrak{A}^4(x) - 15A^4(x)\right) + \left(4\mathfrak{A}^3(x) - 15A^3(x)\right) - 15A^2(x) + B^2(x) + B^0 + C^1(x) + D^0,$$
$$f_5(x) = 720A^6(x) + 24\mathfrak{A}^4(x) + 8\mathfrak{A}^2(x) - 2B^2(x) - 2D^0,$$

where $\mathfrak{A}^0, A^0, B^0, C^0, D^0$ are arbitrary constants, $\mathfrak{A}^n(x), A^n(x), B^n(x), C^n(x)$ are the diagonals of n-additive symmetric functions $\mathfrak{A}_n, A_n, B_n, C_n$, respectively, $n=1,2,\ldots,6$. This completes the proof Theorem 2.2. ❐

3. Corollaries

In this section, we apply our main theorem to derive almost all of the above – mentioned previous results. The first corollary is an extension of Hengkrawit and Thanyacharoen's result (2012) corresponding to (1.12).

Corollary 3.1 The function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation

(1.12) $f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)=13\big(f(x+y)+f(x-y)\big)+168f(y)$

for all $x,y\in\mathbb{R}$ if and only if it is of the form $f(x)=A^4(x)$ for all $x\in\mathbb{R}$, where $A^4(x)$ is the diagonal of 4-additive symmetric function $A_4:\mathbb{R}^4\to\mathbb{R}$.

Proof. By Theorem 2.2, taking $f_1(x)=f_2(x)=f(x), f_3(x)=22f(x), f_4(x)=13f(x)$ and $f_5(x)=168f(x)$. Next, using Maple, which is the popular mathematical software packet, produces the reduced row echelon form, we have

(3.1) $$f(x)=\frac{1}{7}A^4(x)+\frac{1}{7}A^3(x)+\frac{15}{28}A^1(x)+\frac{15}{28}A^0(x)+\frac{1}{28}B^2(x)+\frac{1}{28}B^0(x)+\frac{1}{28}C^1(x).$$

Now, substituting (3.1) into (1.12), we get

(3.2) $$\begin{aligned}&\frac{26}{7}\mathfrak{A}^4(x)+\frac{156}{7}\mathfrak{A}^{2,2}(x,y)+\frac{194}{7}\mathfrak{A}^4(y)+\frac{26}{7}\mathfrak{A}^3(x)\\&+\frac{78}{7}\mathfrak{A}^{1,2}(x,y)+\frac{390}{28}\mathfrak{A}^1(x)+\frac{390}{28}\mathfrak{A}^0(x)+\frac{26}{28}B^2(x)\\&+\frac{26}{28}B^2(y)+\frac{26}{28}B^0(x)+\frac{26}{28}C^1(x)\\&=\frac{26}{7}\mathfrak{A}^4(x)+\frac{156}{7}\mathfrak{A}^{2,2}(x,y)+\frac{194}{7}\mathfrak{A}^4(y)+\frac{26}{7}\mathfrak{A}^3(x)\\&+\frac{78}{7}\mathfrak{A}^{1,2}(x,y)+24\mathfrak{A}^3(y)+\frac{390}{28}\mathfrak{A}^1(x)+90\mathfrak{A}^1(y)\\&+\frac{390}{28}\mathfrak{A}^0(x)+90\mathfrak{A}^0(y)+\frac{26}{28}B^2(x)+\frac{26}{28}B^2(y)+6B^2(y)\\&+\frac{26}{28}B^0(x)+6B^0(y)+\frac{26}{28}C^1(x)+6C^1(y).\end{aligned}$$

Thus

(3.3) $24\mathfrak{A}^3(y)+90\mathfrak{A}^1(y)+90\mathfrak{A}^0(y)+6B^2(y)+6B^0(y)+6C^1(y)=0$, and then dividing (3.3) by 168, we get

(3.4) $$\frac{1}{7}\mathfrak{A}^3(x)+\frac{15}{28}\mathfrak{A}^1(x)+\frac{15}{28}\mathfrak{A}^0(x)+\frac{1}{28}B^2(x)+\frac{1}{28}B^0(x)+\frac{1}{28}C^1(x)=0.$$

Substituting (3.4) into (3.1), we get

(3.5) $$f(x)=\frac{1}{4}\mathfrak{A}^4(x).$$

By Proposition 1.2, we get

(3.6) $$f(x)=\mathfrak{A}^4\Big(\sqrt[4]{\tfrac{1}{4}}x\Big).$$

Putting $\sqrt[4]{\frac{1}{4}}x=x$ in (3.6), we get $f(x)=\mathfrak{A}^4(x)$ ❐

The next corollary is an extension of Chung and Sahoo's result (2003) correspond-ding to (1.8).

Corollary 3.2 The function $f:\mathbb{R}\to\mathbb{R}$ satisfies the quartic functional equation

(1.8) $f(x+2y)+f(x-2y)+6f(x)= =4\big(f(x+y)+f(x-y)\big)+24f(y)$ for all $x,y\in\mathbb{R}$, if and only if it is of the form $f(x)=A^4(x)$ for all $x\in\mathbb{R}$, where $A^4(x)$ is the diagonal of 4-additive symmetric function $A_4:\mathbb{R}^4\to\mathbb{R}$.

Proof. By Theorem 2.2, taking $f_1(x)=0$, $f_2(x)=f(x)$, $f_3(x)=6f(x)$, $f_4(x)=4f(x)$ and $f_5(x)=24f(x)$ and using Maple, we have

(3.7) $f(x)=\mathfrak{A}^4(x)+\mathfrak{A}^3(x)+\frac{15}{4}\mathfrak{A}^1(x)+\frac{15}{4}\mathfrak{A}^0(x) +\frac{1}{4}B^2(x)+\frac{1}{4}B^0(x)+\frac{1}{4}C^1(x).$

Substituting (3.7) into (1.8) we get

$$8\mathfrak{A}^4(x)+48\mathfrak{A}^{2,2}(x,y)+32\mathfrak{A}^4(y)+8\mathfrak{A}^3(x)+24\mathfrak{A}^{1,2}(x,y) +30\mathfrak{A}^1(x)+30\mathfrak{A}^0(x)+2B^2(y)+2B^0(x)+2C^1(x) =8\mathfrak{A}^4(x)+48\mathfrak{A}^{2,2}(x,y)+32\mathfrak{A}^4(y)+8\mathfrak{A}^3(x) +24\mathfrak{A}^{1,2}(x,y)+30\mathfrak{A}^1(x)+30\mathfrak{A}^0(x)+2B^2(y) +2B^0(x)+2C^1(x)+24\mathfrak{A}^3(y)+90\mathfrak{A}^1(y)+90\mathfrak{A}^0(x) +6B^2(y)+6B^0(y)+6C^1(y).$$

Thus

(3.8) $24\mathfrak{A}^3(y)+90\mathfrak{A}^1(y)+90\mathfrak{A}^0(x)+6B^2(y) +6B^0(y)+6C^1(y)=0,$ then dividing (3.8) by 24, we get

(3.9) $\mathfrak{A}^3(x)+\frac{15}{4}\mathfrak{A}^1(x)+\frac{15}{4}\mathfrak{A}^0(x)+\frac{1}{4}B^2(x) +\frac{1}{4}B^0(x)+\frac{1}{4}C^1(x)=0.$

Substituting (3.9) into (3.7), we get $f(x)=\mathfrak{A}^4(x).$ ❒

Taking $f_1(x)=f_5(x)=0$, $f_2(x)=f(x)$, $f_3(x)=6f(x)$ and $f_4(x)=4f(x)$, Theorem 2.2 yields an extension Sahoo's result (2003) corresponding to (1.9).

Corollary 3.3 The function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation

(1.9) $f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)$ for all $x,y\in\mathbb{R}$ if and only if it is of the form $f(x)=A^3(x)+A^2(x)+A^1(x)+A^0(x)$ for all $x\in\mathbb{R}$, where A^0 are arbitrary constants, $A^n(x)$ are the diagonals of n-additive symmetric functions A_n, respectively, $n=1,2,3$.

Proof. Using Maple, we have

(3.10) $f(x)=\mathfrak{A}^4(x)+\mathfrak{A}^3(x)+\frac{15}{4}\mathfrak{A}^1(x)+\frac{15}{4}\mathfrak{A}^0(x) +\frac{1}{4}B^2(x)+\frac{1}{4}B^0(x)+\frac{1}{4}C^1(x).$

Substituting (3.10) into (1.9), we get

$$8\mathfrak{A}^4(x)+48\mathfrak{A}^{2,2}(x,y)+32\mathfrak{A}^4(y)+8\mathfrak{A}^3(x) +24\mathfrak{A}^{1,2}(x,y)+30\mathfrak{A}^1(x)+30\mathfrak{A}^0(x)+2B^2(x) +2B^2(y)+2B^0(x)+2C^1(x)=8\mathfrak{A}^4(x)+48\mathfrak{A}^{2,2}(x,y) +8\mathfrak{A}^4(y)+8\mathfrak{A}^3(x)+24\mathfrak{A}^{1,2}(x,y)+30\mathfrak{A}^1(x)+30\mathfrak{A}^0(x) +2B^2(y)+2B^0(x)+2C^1(x).$$

Thus

(3.11) $24\mathfrak{A}^4(y)=0,$

dividing (3.11) by 24, we get

(3.12) $\mathfrak{A}^4(x)=0.$

Substituting (3.12) into (3.10) we get $f(x)=E^3(x)+E^2(x)+E^1(x)+E^0$, where $E^2(x)=\frac{1}{4}B^2(x), E^1(x)=\frac{15}{4}\mathfrak{A}^1(x)+\frac{1}{4}C^1(x)$ and $E^0=\frac{15}{4}\mathfrak{A}^0+\frac{1}{4}B^0$, E^0 are arbitrary constants, $E^n(x)$ are the diagonals of n-additive symmetric functions E_n, respectively, $n=1,2,3$. ❒

4. Conclusion

A general solution of the functional equation $f_1(x+3y)+f_1(x-3y)+f_2(x+2y)+ f_2(x-2y)+f_3(x)=f_4(x+y)+f_4(x-y)+f_5(y)$

are given by $f_1(x) = A^6(x) + A^5(x) + A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0$, $f_2(x) = -6A^6(x) - 6A^5(x) + \left(\mathfrak{A}^4(x) - 6A^4(x)\right) + \left(\mathfrak{A}^3(x) - 6A^3(x)\right) + \left(\mathfrak{A}^2(x) - 6A^2(x)\right) + \mathfrak{A}^1(x) + \mathfrak{A}^0$, $f_3(x) = -20A^6(x) - 20A^5(x) + \left(6\mathfrak{A}^4(x) - 20A^4(x)\right) + \left(6\mathfrak{A}^3(x) - 20A^3(x)\right) - \left(2\mathfrak{A}^2(x) + 20A^2(x)\right) - 2\left(\mathfrak{A}^1(x) + A^1(x)\right) - 2\left(\mathfrak{A}^0 + A^0\right) + 2B^2(x) + 2C^1(x) + 2B^0$, $f_4(x) = -15A^6(x) - 15A^5(x) + \left(4\mathfrak{A}^4(x) - 15A^4(x)\right) + \left(4\mathfrak{A}^4(x) - 15A^4(x)\right) + \left(4\mathfrak{A}^3(x) - 15A^3(x)\right) - 15A^2(x) + B^2(x) + B^0 + C^1(x) + D^0$, $f_5(x) = 720A^6(x) + 24\mathfrak{A}^4(x) + 8\mathfrak{A}^2(x) - 2B^2(x) - 2D^0$, where $\mathfrak{A}^0, A^0, B^0, C^0, D^0$ are arbitrary constants, $\mathfrak{A}^n(x), A^n(x), B^n(x), C^n(x)$ are the diagonals of n-additive symmetric functions $\mathfrak{A}_n, A_n, B_n, C_n$, respectively, $n = 1, 2, \ldots, 6$.

Specializing the functions involved, three related results extending a number of previously known general solutions, such as the ones dealing with the quartic functional equation and the polynomial functional equation degree three, are derived as direct consequences by using Maple for producing the reduced row echelon form.

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