Irrational Numbers with Generalized Bounded Partial Quotients

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Abstract

The definition of irrational numbers with generalized bounded partial quotients is defined. We will investigate some relation between irrational numbers with bounded partial quotients in simple continued fraction and in generalized continued fraction. Furthermore, using three distance theorem of Slater, we also find the result of three distance theorem for generalized continued fraction in some kind of sequence.

Keywords: continued fraction; bounded partial quotients; three distance theorem.

1. Introduction

A continued fraction is an expression of the form $\alpha_0 + \cfrac{\beta_1}{\alpha_1 + \cfrac{\beta_2}{\alpha_2 + \ddots}}$, where α_n and β_n may be any kind of numbers, variables, or functions. The number of term can be either finite or infinite. If $\alpha_0 \in \mathbb{Z}$, $\alpha_n \in \mathbb{N}$ and $\beta_n = 1$ for all $n \geq 1$, then the expression is called *a simple continued fraction*; otherwise, it is called *a generalized continued fraction*. For simplicity, we denote the term $\cfrac{\beta_1}{\alpha_1 + \cfrac{\beta_2}{\alpha_2 + \ddots}}$ by $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{\beta_n}{\alpha_n}$.

It is also known that all irrational numbers can be uniquely expressed as a simple continued fraction. Moreover, simple continued fractions expansion of any real number is finite if and only if that number is rational.

For examples, $\frac{19}{8} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2}}}$,

$\frac{13}{95} = \cfrac{1}{7 + \cfrac{1}{3 + \cfrac{1}{4}}}$, $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}$,

and $\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \ddots}}}$.

Thus, we can determine that a fixed irrational number θ has infinite simple continued fraction expansion:

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$$\theta = a_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{a_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}},$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for all $n \geq 1$. We start with introducing some definition about simple continued fraction.

Definition 1.1 We say that θ has bounded partial quotients if and only if $\sup_{n\geq 1} a_n < \infty$. ❐

Definition 1.2 Define integers p_n and q_n by: $p_{-1} = 1$, $p_0 = a_0$, $p_n = a_n p_{n-1} + p_{n-2}$ for $n \geq 1$, $q_{-1} = 0$, $q_0 = 1$, $q_n = a_n q_{n-1} + q_{n-2}$ for $n \geq 1$ and define $\eta_n = |q_n \theta - p_n|$ for all $n \geq -1$. ❐

Remark*:* For all $n \geq 1$, $\frac{p_n}{q_n}$ is called n$^{\text{th}}$ convergent to θ and satisfy $\frac{p_n}{q_n} = a_0 + \underset{i=1}{\overset{n}{\mathrm{K}}} \frac{1}{a_i}$; in fact, $\lim_{n\to\infty} \frac{p_n}{q_n} = \theta$.

Definition 1.3 For any real number x, let $\lfloor x \rfloor$ denote the largest integer less than or equal to x and let $\{x\}$ denote the fractional part of x, i.e., $\{x\} = x - \lfloor x \rfloor$.

For $N \in \mathbb{N}$, consider the sequence of distinct points $\{1\theta\}, \{2\theta\}, ..., \{N\theta\}$ in $[0,1]$, arranged in increasing order: $0 < \{k_1\theta\} < \{k_2\theta\} < \cdots < \{k_N\theta\} < 1$, where $k_j \in \{1,2,...,N\}$ for $j = 1,2,...,N$. Then, interval $[0,1]$ is partitioned into $N+1$ subintervals. A set of these subintervals are a partition of $[0,1]$. Let $P_\theta(N)$ denote an aforementioned partition. The maximum and minimum lengths of the subintervals of $P_\theta(N)$ are denoted, respectively, by $d_\theta(N)$ and $d'_\theta(N)$. ❐

In 1998, Mukherjee and Karner proved that θ has bounded partial quotients if and only if the sequence $\left(\frac{d_\theta(N)}{d'_\theta(N)}\right)_{N=1}^{\infty}$ is bounded by using a theorem proved by Slater (1967) as follows:

Theorem 1.4 (three distance theorem)

(I) For all $N \in \mathbb{N}$, N can be uniquely represented as $N = rq_k + q_{k-1} + s$, for uniquely integer $k \geq 0$, $1 \leq r \leq a_{k+1}$, and $0 \leq s < q_k$.

(II) For the partition $P_\theta(N)$, represent N as $N = rq_k + q_{k-1} + s$ in part (I), then there are $(r-1)q_k + q_{k-1} + s + 1$ subintervals of length η_k, $s+1$ subintervals of length $\eta_{k-1} - r\eta_k$, and $q_k - (s+1)$ subintervals of length $\eta_{k-1} - (r-1)\eta_k$. ❐

Because θ can be expressed as an infinite generalized continued fraction:

$$\theta = \alpha_0 + \underset{n=1}{\overset{\infty}{\mathrm{K}}} \frac{\beta_n}{\alpha_n} = \alpha_0 + \cfrac{\beta_1}{\alpha_1 + \cfrac{\beta_2}{\alpha_2 + \ddots}},$$

the definition of irrational numbers with bounded partial quotients may be established. In this paper, we attend in case that α_0 is an integer, α_n and β_n are positive integer for all $n \geq 1$. However, for a given sequence $(\beta_n)_{n=1}^{\infty}$ of positive integer, θ may have more than one generalized continued fraction expansion. For example, let $\theta = \sqrt{2}$, $\beta_n = 3$ for all $n \geq 1$, we get

$$\sqrt{2} = 1 + \cfrac{3}{6 + \cfrac{3}{2 + \cfrac{3}{6 + \cfrac{3}{2 + \ddots}}}}, \quad \sqrt{2} = 1 + \cfrac{3}{7 + \cfrac{3}{12 + \cfrac{3}{8 + \cfrac{3}{12 + \ddots}}}},$$

$$\sqrt{2}=1+\cfrac{3}{5+\cfrac{3}{1+\cfrac{3}{8+\cfrac{3}{3+\ddots}}}}, \text{ etc.}$$

Hence, we must obviate this ambiguousness for aforementioned purpose.

2. Numbers with generalized bounded partial quotients

In this section, we provide the definition of numbers with bounded partial quotients in generalized continued fraction form and investigate some basic property related with numbers of simple continued fraction form. First of all, we start with proving two lemmas.

Lemma 2.1 For any sequences $(\alpha_n)_{n=0}^{\infty}$, $(\beta_n)_{n=1}^{\infty}$ of complex number, where $\alpha_n,\beta_n\neq 0$ for all $n\geq 1$. If we define complex numbers P_n and Q_n by: $P_{-1}=1$, $P_0=\alpha_0$, $P_n=\alpha_nP_{n-1}+\beta_nP_{n-2}$ for $n\geq 1$, $Q_{-1}=0$, $Q_0=1$, $Q_n=\alpha_nQ_{n-1}+\beta_nQ_{n-2}$ for $n\geq 1$, then $\frac{P_n}{Q_n}=\alpha_0+\operatorname*{K}_{j=1}^{n}\frac{\beta_j}{\alpha_j}$ and $Q_nP_{n-1}-Q_{n-1}P_n$ $=(-1)^n\beta_1\beta_2\cdots\beta_n$ for all $n\geq 1$.

Proof. For any sequences $(\alpha_n)_{n=0}^{\infty},(\beta_n)_{n=1}^{\infty}$ of complex number. We see that $\frac{P_1}{Q_1}=\alpha_0+\frac{\beta_1}{\alpha_1}$ and $Q_1P_0-Q_0P_1=-\beta_1$. By mathematical induction, assume that $\frac{P_n}{Q_n}=\alpha_0+\operatorname*{K}_{j=1}^{n}\frac{\beta_j}{\alpha_j}$ and $Q_nP_{n-1}-Q_{n-1}P_n=$ $(-1)^n\beta_1\beta_2\cdots\beta_n$, we obtain that $Q_{n+1}P_n-Q_nP_{n+1}=$ $-\beta_{n+1}(Q_nP_{n-1}-Q_{n-1}P_n)=(-1)^{n+1}\beta_1\beta_2\cdots\beta_{n+1}$

Since $\frac{\alpha_nP_{n-1}+\beta_nP_{n-2}}{\alpha_nQ_{n-1}+\beta_nQ_{n-2}}=\frac{P_n}{Q_n}=\alpha_0+\operatorname*{K}_{j=1}^{n}\frac{\beta_j}{\alpha_j}$, replace the term α_n by $\alpha_n+\frac{\beta_{n+1}}{\alpha_{n+1}}$ we obtain

$$\alpha_0+\operatorname*{K}_{j=1}^{n+1}\frac{\beta_j}{\alpha_j}=\frac{\left(\alpha_n+\frac{\beta_{n+1}}{\alpha_{n+1}}\right)P_{n-1}+\beta_nP_{n-2}}{\left(\alpha_n+\frac{\beta_{n+1}}{\alpha_{n+1}}\right)Q_{n-1}+\beta_nQ_{n-2}}=\frac{P_{n+1}}{Q_{n+1}}.$$

We conclude that $\frac{P_n}{Q_n}=\alpha_0+\operatorname*{K}_{j=1}^{n}\frac{\beta_j}{\alpha_j}$ and $Q_nP_{n-1}-Q_{n-1}P_n=(-1)^n\beta_1\beta_2\cdots\beta_n$ for all $n\geq 1$. ❒

Lemma 2.2 For a given sequence $(B_n)_{n=1}^{\infty}$ of positive integer, θ can be uniquely written as $\theta=A_0+\operatorname*{K}_{n=1}^{\infty}\frac{B_n}{A_n}=A_0+\cfrac{B_1}{A_1+\cfrac{B_2}{A_2+\ddots}}$, where $A_0\in\mathbb{Z}$, $A_n\in\mathbb{N}$ and $0<\operatorname*{K}_{j=n}^{\infty}\frac{B_j}{A_j}<1$ for all $n\geq 1$.

Proof. Let $A_0=\lfloor\theta\rfloor$, $\theta_0=\{\theta\}$; $A_n=\left\lfloor\frac{B_n}{\theta_{n-1}}\right\rfloor$, $\theta_n=\left\{\frac{B_n}{\theta_{n-1}}\right\}$ for all $n\geq 1$. We see that A_0 is an integer, A_n are positive integer, $0<\theta_{n-1}<1$ and $B_n\leq A_n$ for all $n\geq 1$. By mathematical induction, we have

$$\theta=A_0+\cfrac{B_1}{A_1+\cfrac{B_2}{A_2+\ddots\cfrac{B_n}{A_n+\theta_n}}},\quad \text{for all } n\geq 1.$$

We can define integers P_n and Q_n as in Lemma 2.1, then we have $A_0+\operatorname*{K}_{j=1}^{n}\frac{B_j}{A_j}=\frac{P_n}{Q_n}=\frac{A_nP_{n-1}+B_nP_{n-2}}{A_nQ_{n-1}+B_nQ_{n-2}}$. By replacing A_n by $A_n+\theta_n$, we obtain $\theta=\frac{(A_n+\theta_n)P_{n-1}+B_nP_{n-2}}{(A_n+\theta_n)Q_{n-1}+B_nQ_{n-2}}$ $=\frac{P_n+\theta_nP_{n-1}}{Q_n+\theta_nQ_{n-1}}$, for all $n\geq 1$. By Lemma 2.1, we get $\left|\theta-\frac{P_n}{Q_n}\right|=\left|\frac{P_n+\theta_nP_{n-1}}{Q_n+\theta_nQ_{n-1}}-\frac{P_n}{Q_n}\right|=\left|\frac{\theta_n\left(Q_nP_{n-1}-Q_{n-1}P_n\right)}{Q_n(Q_n+\theta_nQ_{n-1})}\right|$

$=\dfrac{B_1B_2\cdots B_n\theta_n}{Q_n(Q_n+\theta_nQ_{n-1})}$. By mathematical induction, we have $0<\dfrac{B_1B_2\cdots B_n\theta_n}{Q_n}<1$ for all $n\geq 1$. Since $\lim\limits_{n\to\infty}\dfrac{1}{Q_n+\theta_nQ_{n-1}}=0$ and by the squeeze theorem, we have $\lim\limits_{n\to\infty}\left|\theta-\dfrac{P_n}{Q_n}\right|=0$. Thus, $\theta=\lim\limits_{n\to\infty}\dfrac{P_n}{Q_n}=\lim\limits_{n\to\infty}\left(A_0+\operatorname*{K}_{j=1}^{n}\dfrac{B_j}{A_j}\right)$,i.e., $\theta=A_0+\cfrac{B_1}{A_1+\cfrac{B_2}{A_2+\ddots}}$.

. By mathematical induction, we obtain $0<\theta_{n-1}=\operatorname*{K}_{j=n}^{\infty}\dfrac{B_j}{A_j}<1$ for all $n\geq 1$. Hence, θ can be written as $\theta=A_0+\operatorname*{K}_{n=1}^{\infty}\dfrac{B_n}{A_n}$, where $0<\operatorname*{K}_{j=n}^{\infty}\dfrac{B_j}{A_j}<1$ for all $n\geq 1$. For uniqueness, let $\theta=C_0+\operatorname*{K}_{n=1}^{\infty}\dfrac{B_n}{C_n}$, where $C_0\in\mathbb{Z}$, $C_n\in\mathbb{N}$ and $0<\operatorname*{K}_{j=n}^{\infty}\dfrac{B_j}{C_j}<1$ for all $n\geq 1$. Since C_0 is an integer and $0<\operatorname*{K}_{n=1}^{\infty}\dfrac{B_n}{C_n}<1$, we have $C_0=\lfloor\theta\rfloor=A_0$ and $\operatorname*{K}_{n=1}^{\infty}\dfrac{B_n}{C_n}=\theta_0$. By mathematical induction, assume that $C_n=A_n$ and $\operatorname*{K}_{j=n+1}^{\infty}\dfrac{B_j}{C_j}=\theta_n$. Then $C_{n+1}+\operatorname*{K}_{j=n+2}^{\infty}\dfrac{B_j}{C_j}=\dfrac{B_{n+1}}{\theta_n}$. Since C_{n+1} is an integer and $0<\operatorname*{K}_{j=n+2}^{\infty}\dfrac{B_j}{C_j}<1$, we obtain $C_{n+1}=\left\lfloor\dfrac{B_{n+1}}{\theta_n}\right\rfloor=A_{n+1}$ and $\operatorname*{K}_{j=n+2}^{\infty}\dfrac{B_j}{C_j}=\left\{\dfrac{B_{n+1}}{\theta_n}\right\}=\theta_0$. Hence, $C_n=A_n$ for all $n\geq 0$. ❐

By Lemma 2.2, let $(B_n)_{n=1}^{\infty}$ be a fixed sequence of positive integer, we can write θ as a form in the Lemma. So we have $\theta=a_0+\operatorname*{K}_{n=1}^{\infty}\dfrac{1}{a_n}=A_0+\operatorname*{K}_{n=1}^{\infty}\dfrac{B_n}{A_n}$, where $a_0\in\mathbb{Z}$, $A_0\in\mathbb{Z}$, $a_n\in\mathbb{N}$, $A_n\in\mathbb{N}$ and $0<\operatorname*{K}_{j=n}^{\infty}\dfrac{B_j}{A_j}<1$ for all $n\geq 1$. Then there is the definition of n[th] convergent to θ for generalized continued fraction which as in Lemma 2.1.

Definition 2.5 Define integers P_n and Q_n by: $P_{-1}=1$, $P_0=A_0$, $P_n=A_nP_{n-1}+B_nP_{n-2}$ for $n\geq 1$, $Q_{-1}=0$, $Q_0=1$, $Q_n=A_nQ_{n-1}+B_nQ_{n-2}$ for $n\geq 1$ and define $\mu_n=|Q_n\theta-P_n|$ for all $n\geq -1$. ❐

Now we can define numbers with bounded partial quotients in generalized continued fraction form and get some preliminary fact as follows:

Definition 2.3 We say that θ has bounded $(B_n)_{n=1}^{\infty}$'s partial quotients if and only if $\sup\limits_{n\geq 1}A_n<\infty$. ❐

Theorem 2.4 For all $n\geq 1$, $B_n\leq A_n$ and if the sequence $(B_n)_{n=1}^{\infty}$ is not bounded, then θ has not bounded $(B_n)_{n=1}^{\infty}$'s partial quotients.

Proof. Let $n\geq 1$. Since $\operatorname*{K}_{j=n}^{\infty}\dfrac{B_j}{A_j}<1$ and $\operatorname*{K}_{j=n+1}^{\infty}\dfrac{B_j}{A_j}<1$, $B_n<A_n+\operatorname*{K}_{j=n+1}^{\infty}\dfrac{B_j}{A_j}<A_n+1$. Since A_n and B_n are integer, $B_n\leq A_n$. Hence, if the sequence $(B_n)_{n=1}^{\infty}$ is not bounded, then $(A_n)_{n=1}^{\infty}$ is not bounded i.e. θ has not bounded $(B_n)_{n=1}^{\infty}$'s partial quotients. ❐

If θ has bounded partial quotients. We want to find necessary conditions and sufficient conditions for sequence $(B_n)_{n=1}^{\infty}$ of positive integer such that θ also has bounded $(B_n)_{n=1}^{\infty}$'s partial quotients. Theorem 2.4 give we a necessary condition. In this paper, we provide a basic example of sufficient condition for $(B_n)_{n=1}^{\infty}$ such that θ has bounded $(B_n)_{n=1}^{\infty}$'s partial quotients.

From now on we attend in case that the sequence $(B_n)_{n=1}^{\infty}$ satisfying $B_{2n-1}=B_{2n}\le a_{2n}$ for all $n\ge 1$. Before reach to the next lemma, for simplicity, we define $f(n)=\frac{1-(-1)^n}{2}=\begin{cases}0 & ;\text{ n is even}\\ 1 & ;\text{ n is odd}\end{cases}$ for all integer n.

Lemma 2.6 For all $n\ge 0$,

$$A_n=\begin{cases}a_n & ;\text{ n is even}\\ B_n a_n & ;\text{ n is odd}\end{cases}=\left(f(n)(B_n-1)+1\right)a_n .$$

Proof. For $n=0$, we have $a_0=\lfloor\theta\rfloor=A_0$. Furthermore, we see that $f(n)(B_n-1)+1=\begin{cases}1 & ;\text{ n is even}\\ B_n & ;\text{ n is odd}\end{cases}$ for all $n\ge 1$. Consider $\theta=a_0+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\frac{1}{a_n}$, for all $n\ge 1$, we multiply the term $\underset{i=2n-1}{\overset{\infty}{\mathrm{K}}}\frac{1}{a_i}$ by $\frac{B_{2n-1}}{B_{2n}}=1$. Then we obtain

$$\theta=a_0+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\frac{1}{a_n}=A_0+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\frac{B_n}{\left(f(n)(B_n-1)+1\right)a_n}.$$

Since $B_{2n-1}=B_{2n}\le a_{2n}$ for all $n\ge 1$, $0<\underset{j=n}{\overset{\infty}{\mathrm{K}}}\frac{B_j}{\left(f(j)(B_j-1)+1\right)a_j}<1$ for all $n\ge 1$. But θ is uniquely written as $\theta=A_0+\underset{n=1}{\overset{\infty}{\mathrm{K}}}\frac{B_n}{A_n}$, where $0<\underset{j=n}{\overset{\infty}{\mathrm{K}}}\frac{B_j}{A_j}<1$ for all $n\ge 1$. Hence, $A_n=\left(f(n)(B_n-1)+1\right)a_n$ for all $n\ge 0$. ❒

Theorem 2.7. θ has bounded partial quotients if and only if θ has bounded $(B_n)_{n=1}^{\infty}$'s partial quotients.

Proof. First, we will show that if θ has bounded partial quotients, then θ has bounded $(B_n)_{n=1}^{\infty}$'s partial quotients. Assume that θ has bounded partial quotients. Then there exists $a\in\mathbb{R}$ such that $a_n\le a$ for all $n\ge 1$. By Lemma 2.6, we get $A_n\le B_n a_n\le a^2$ for all $n\ge 1$, i.e., θ has bounded $(B_n)_{n=1}^{\infty}$'s partial quotients. Conversely, assume that θ has bounded $(B_n)_{n=1}^{\infty}$'s partial quotients. Then there exists $A\in\mathbb{R}$ such that $A_n\le A$, for all $n\ge 1$. By Lemma 2.6, $a_n\le A_n\le A$ for all $n\ge 1$ i.e. θ has bounded partial quotients. ❒

Lemma 2.8 For all $n\ge -1$,

$$Q_n=\begin{cases}B_0B_2\cdots B_n q_n & ;\text{ n is even}\\ B_0B_2\cdots B_{n+1}q_n & ;\text{ n is odd}\end{cases}=B_0B_2\cdots B_{n+f(n)}q_n,$$

$$P_n=\begin{cases}B_0B_2\cdots B_n p_n & ;\text{ n is even}\\ B_0B_2\cdots B_{n+1}p_n & ;\text{ n is odd}\end{cases}=B_0B_2\cdots B_{n+f(n)}p_n,$$

$$\mu_n=\begin{cases}B_0B_2\cdots B_n \eta_n & ;\text{ n is even}\\ B_0B_2\cdots B_{n+1}\eta_n & ;\text{ n is odd}\end{cases}=B_0B_2\cdots B_{n+f(n)}\eta_n,$$

where $B_0=1$.

Proof. First, we have $Q_{-1}=B_0q_{-1}$ and $Q_0=B_0q_0$. By mathematical induction, assume that $Q_n=\begin{cases}B_0B_2\cdots B_n q_n & ;\text{ n is even}\\ B_0B_2\cdots B_{n+1}q_n & ;\text{ n is odd}\end{cases}=B_0B_2\cdots B_{n+f(n)}q_n$ and $Q_{n-1}=\begin{cases}B_0B_2\cdots B_{n-1}q_{n-1} & ;\text{ n-1 is even}\\ B_0B_2\cdots B_n q_{n-1} & ;\text{ n-1 is odd}\end{cases}=B_0B_2\cdots B_{n-1+f(n-1)}q_{n-1}$.

- If n is even, then $Q_{n+1}=A_{n+1}Q_n+B_{n+1}Q_{n-1}$. By Lemma 2.6, we have $Q_{n+1}=B_{n+1}a_{n+1}Q_n+B_{n+1}Q_{n-1}$ $=B_{n+1}a_{n+1}(B_0B_2\cdots B_nq_n)+B_{n+1}(B_0B_2\cdots B_nq_{n-1})$ $=B_0B_2\cdots B_{n+2}(a_{n+1}q_n+q_{n-1})$; since $n+1$ is odd, $B_{n+1}=B_{n+2}$, $=B_0B_2\cdots B_{n+2}q_{n+1}$

- If n is odd, then $Q_{n+1} = A_{n+1}Q_n + B_{n+1}Q_{n-1}$. By Lemma 2.6, we have $Q_{n+1} = a_{n+1}Q_n + B_{n+1}Q_{n-1}$ $= a_{n+1}(B_0B_2\cdots B_{n+1}q_n) + B_{n+1}(B_0B_2\cdots B_{n-1}q_{n-1})$ $= B_0B_2\cdots B_{n+1}(a_{n+1}q_n + q_{n-1}) = B_0B_2\cdots B_{n+1}q_{n+1}$

  Hence, $Q_{n+1} = \begin{cases} B_0B_2\cdots B_{n+1}q_{n+1} & ;\ \text{n+1 is even} \\ B_0B_2\cdots B_{n+2}q_{n+1} & ;\ \text{n+1 is odd} \end{cases}$

  $= B_0B_2B_4\cdots B_{n+1+f(n+1)}q_{n+1}$. We conclude that $Q_n = B_0B_2B_4\cdots B_{n+f(n)}q_n$ for all $n \geq -1$. Similarly, we obtain

  $P_n = B_0B_2B_4\cdots B_{n+f(n)}p_n$ and $\mu_n = B_0B_2B_4\cdots B_{n+f(n)}\eta_n$ for all $n \geq -1$. ❐

Finally, we get the result of three distance theorem for generalized continued fraction, which is our main result, for the sequence $(B_n)_{n=1}^{\infty}$ which satisfied $B_{2n-1} = B_{2n} \leq a_{2n}$ for all $n \geq 1$.

Theorem 2.9 (three distance theorem for generalized continued fraction).

(I) For all $N \in \mathbb{N}$, N can be uniquely represented as $N = r\frac{Q_k}{B_0B_2\cdots B_{k+f(k)}} + \frac{Q_{k-1}}{B_0B_2\cdots B_{k-1+f(k-1)}} + s$, for uniquely integer $k \geq 0$, $1 \leq r \leq \frac{A_{k+1}}{f(k+1)(B_{k+1}-1)+1}$, and $0 \leq s < \frac{Q_k}{B_0B_2\cdots B_{k+f(k)}}$.

(II) For the partition $P_\theta(N)$, represent N as $N = r\frac{Q_k}{B_0B_2\cdots B_{k+f(k)}} + \frac{Q_{k-1}}{B_0B_2\cdots B_{k-1+f(k-1)}} + s$ in part (I), then there are $(r-1)\frac{Q_k}{B_0B_2\cdots B_{k+f(k)}} + \frac{Q_{k-1}}{B_0B_2\cdots B_{k-1+f(k-1)}} + s + 1$ subintervals of length $\frac{\mu_k}{B_0B_2\cdots B_{k+f(k)}}$, $s+1$ subintervals of length $\frac{\mu_{k-1}}{B_0B_2\cdots B_{k-1+f(k-1)}} - r\frac{\mu_k}{B_0B_2\cdots B_{k+f(k)}}$, and $\frac{Q_k}{B_0B_2\cdots B_{k+f(k)}} - (s+1)$ subintervals of length $\frac{\mu_{k-1}}{B_0B_2\cdots B_{k-1+f(k-1)}} - (r-1)\frac{\mu_k}{B_0B_2\cdots B_{k+f(k)}}$.

Proof. The proof of this theorem follows straight from Theorem 1.4 and Lemma 2.8. ❐

3. References

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