Characterization of the Minimizing Graphs with n Vertices and k Cut Edges where $\leq n \leq 12$ and $1 \leq k \leq \frac{n}{2}$

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Abstract

We characterize graphs whose the least eigenvalue attains minimum among all connected graphs of n vertices and k cut edges where $4 \leq n \leq 12$ and $1 \leq k \leq \frac{n}{2}$.

Keywords: adjacency matrix; least eigenvalue; cut edge

1. Introduction

Let G be a simple graph of order n with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$ and edge set $E(G)$. The *adjacency matrix* of G is defined to be a matrix:

$$A(G) = [a_{ij}],$$

where $a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0 & \text{otherwises.} \end{cases}$

Since $A(G)$ is real and symmetric, its eigenvalues are real. Then we can arrange as: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The ***least eigenvalue*** $\lambda_n(G)$ is denoted by $\lambda_{min}(G)$, and the corresponding eigenvectors are called the ***least vectors*** of G. The eigenvalues of $A(G)$ are called the ***eigenvalues*** of G.

In the past the main work on the least eigenvalue of graphs is related its bound. Recently, the problem of minimizing the least eigenvalues of graphs subject to one or more given parameters has studied. Let $\mathcal{G}_n^k$ be the set of all connected graphs of order n with $k\ (\geq 1)$ cut edges. A graph G is called a ***minimizing graph*** in $\mathcal{G}_n^k$ if its least eigenvalue attains the minimum among all graphs in $\mathcal{G}_n^k$.

Wang and Fan (2012) characterized the minimizing graph in $\mathcal{G}_n^k$ where $n > 12$ and $1 \leq k \leq \frac{n}{2}$. In this paper, we characterize minimizing graphs in $\mathcal{G}_n^k$ where $\mathrm{n} \leq 12$ and $1 \leq k \leq \frac{n}{2}$.

2. Preliminaries

Let $x = (x_1, x_2, x_3, \dots, x_n)^T \in \mathbb{R}^n$ and let G be a simple graph with vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_n\}$. Then we can consider x as a function defined on the vertex set of G, that is, each vertex v_i is mapped to $x_i = x(v_i)$. We call x_i is a value of v_i given by x. We can find that

$$x^T A(G) x = 2 \sum_{uv \in E(G)} x(u)x(v) \quad (1)$$

and x is an eigenvector of G corresponding to an eigenvalue λ if and only if $x \neq 0$ and

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$\lambda x(v) = \sum_{u \in N_G(v)} x(u)$, for each $v \in V(G)$ (2)

where $\mathcal{N}_G(v)$ is the neighborhood of v in G. For an arbitrary unit vector

$x \in \mathbb{R}^n$, $\lambda_{\min}(G) \leq x^T A(G) x$ (3)

with equality if and only if x is a least vector of G.

Next, we will state some theorems that are used in this paper.

Theorem 2.1 (Powers, 1989)

If a graph G has k cut edges, then $\lambda_{min}(G) \geq -\sqrt{k}$.

Theorem 2.2 (Bell *et al.*, 2008)

If G is a graph of order n then $\lambda_{min}(G) \geq -\sqrt{\left\lfloor\frac{n}{2}\right\rfloor \cdot \left\lceil\frac{n}{2}\right\rceil}$, with equality if and only if $G = K_{\left\lfloor\frac{n}{2}\right\rfloor,\left\lceil\frac{n}{2}\right\rceil}$.

Theorem 2.3 (Fan *et al.*, 2008)

Let G_1 and G_2 be two disjoint nontrivial connected graph, and let $\{v_1, v_2\} \subseteq V(G_1), u \in V(G_2)$. Let $G = G_1(v_2) \cdot G_2(u)$ and let $\tilde{G} = G_1(v_1) \cdot G_2(u)$. If there exists a least vector x of G such that $|x(v_1)| \geq |x(v_2)|$, then $\lambda_{min}(\tilde{G}) \leq \lambda_{min}(G)$, with equality if and only if x is a least vector of $\tilde{G}$ and $x(v_1) = x(v_2)$ and $\sum_{w \in \mathcal{N}_{G_2(u)}} x(w) = 0$.

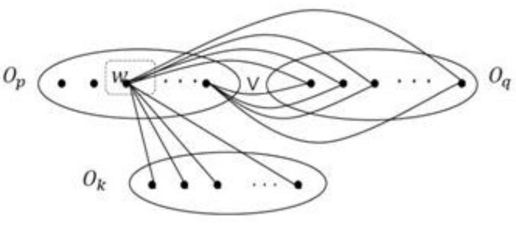

Figure 1 The graph $G(p,q,k)$

Let O_p and O_q be two empty graphs of order p and q, respectively. Let $O_p \vee O_q$ be the graph obtained from $O_p \cup O_q$ together with joining each vertex of O_p to each vertex of O_q. Then $O_p \vee O_q$ is a complete bipartite graph. Let $G(p,q,k)$ denote the graph obtained from $O_p \vee O_q$ by appending k pendant edges to a vertex w of O_p, shown in Figure 1.

Note that if $p \geq 2, q \geq 2$ then $G(p,q,k)$ contains exactly k cut edges; otherwise $G(p,q,k)$ is a tree with $p+q+k-1$ cut edges. We also have $G(p,q,k)$ is bipartite.

Let x be a least vector of $G(p,q,k)$. We have $\lambda_{min}\, G(p,q,k) < 0$ and by (2), x has a constant value x_1 on $V(O_k)$, x_2 on w, x_3 on $V(O_p) - \{w\}$ and x_4 on $V(O_q)$. Thus $\lambda_{min}\, G(p,q,k)$ is the least root of following equations:

$$\begin{aligned} \lambda x_1 &= x_2, \\ \lambda x_2 &= k x_1 + q x_4, \\ \lambda x_3 &= q x_4 \quad and \\ \lambda x_4 &= x_2 + (p-1) x_3. \end{aligned} \quad (4)$$

Then $\lambda_{min}\, G(p,q,k)$ is the least root of following polynomial:

$$f(p,q,k;\lambda) = \lambda^4 - (pq+k)\lambda^2 + k(p-1)q. \quad (5)$$

Moreover, it is easy to see that x contains no zero entries by (4).

Wang and Fan (2012) proved the following result:

Theorem 2.4 (Wang and Fan, 2012)

Among all graphs $G(p,q,k)$, where $p \geq 1, q \geq 1$, $p+q+k = n$, if $n > 12$ and $1 \leq k \leq \frac{n}{2}$, then $G(\lfloor\frac{n-k}{2}\rfloor, \lceil\frac{n-k}{2}\rceil, k)$ is the unique graph whose least eigenvalue attains the minimum.

Note that Theorem 2.4 does not hold for a general n and k. We will consider graphs whose the least eigenvalue attains the minimum for $4 \le n \le 12$ and $1 \le k \le \frac{n}{2}$ by establishing the following lemmas.

Lemma 2.5

Let $4 \le n \le 12$, $1 \le k \le \frac{n}{2}$, $p \ge q+1$ and $n = p+q+k$. Then $\lambda_{min}G(p-1, q+1, k) < \lambda_{min}G(p,q,k)$.

Proof From (5), we have

$$f(p,q,k;\lambda) - f(p-1,q+1,k;\lambda) = (p-q-1)\lambda^2 + k(2+q-p), \quad (6)$$

Thus

$$f(p-1,q+1,k;\lambda_{\min}G(p,q,k)) - \left[(p-q-1)\lambda_{min}^2G(p,q,k) + k(2+q-p)\right] \quad (7)$$

Since $K_{1,k}$ is a proper subgraph of $G(p,q,k)$, we have $\lambda_{\min}G(p,q,k) \le \lambda_{\min}K_{1,k} = -\sqrt{k}$

$$\lambda_{min}^2G(p,q,k) \ge k$$

Since $p \ge q+1$, we have $2+q-p \le 1$. By (7), it implies that

$$(p-1,q+1,k;\lambda_{\min}G(p,q,k)) < 0.$$

Then

$$f(p-1,q+1,k;\lambda_{\min}G(p-1,q+1,k)) > f(p-1,q+1,k;\lambda_{\min}G(p,q,k)).$$

Cleary,when $\lambda \to -\infty$, $f(p-1,q+1,k;\lambda) \to +\infty$. Hence

$\lambda_{\min}G(p-1,q+1,k) < \lambda_{\min}G(p,q,k)$. ❑

Lemma 2.6

Let $4 \le n \le 12$, $1 \le k \le n/2$, $p \le q-3$ and $n = p+q+k$. Then $\lambda_{min}G(p+1, q-1, k) < \lambda_{min}G(p,q,k)$.

Proof From (5), we have $f(p,q,k;\lambda) - f(p+1,q-1,k;\lambda) = (q-p-1)\lambda^2 + k(p-q)$.

Thus $-f\big(p+1,q-1,k;\lambda_{\min}G(p,q,k)\big) =$

$$(q-p-1)\lambda_{min}^2G(p,q,k) + k(p-q). \quad (8)$$

Since $K_{1,k+q}$ is a proper subgraph of $G(p,q,k)$, we have

$\lambda_{\min}G(p,q,k) \le \lambda_{\min}K_{1,k+q} = -\sqrt{k+q}$.

Since $p \le q-3$, we get $q-p-1 \ge 2$ and $q = \frac{n-k+3}{2}$. By (8), we have

$$\begin{aligned} -f\big(p+1,q-1,k;\lambda_{\min}G(p,q,k)\big) &= (q-p-1)\lambda_{min}^2G(p,q,k) + k(p-q) \\ &\ge (q-p-1)(k+q) + k(p-q) \\ &\ge 2\left(\frac{n-k+3}{2}\right) - k \\ &= n-2k+3 > 0. \end{aligned}$$

This implies that $(p+1,q-1,k;\lambda_{\min}G(p+1,q-1,k)) > f(p+1,q-1,k;\lambda_{\min}G(p,q,k))$.Cleary, when $\lambda \to -\infty$, $f(p+1,q-1,k;\lambda) \to +\infty$.

Hence $\lambda_{min}G(p+1, q-1, k) < \lambda_{min}G(p,q,k)$. ❑

Lemma 2.7

Let $4 \le n \le 12$, $1 \le k \le n/2$, $p = q-2$ and $n = p+q+k$.

Then
- $\lambda_{min}\big(G(p+1,q-1,k)\big) \le \lambda_{min}G(p,q,k)$ if (n = 12, k = 6), (n = 8, k = 4) and (n = 7, k = 3),
- $\lambda_{min}\big(G(p+1,q-1,k)\big) < \lambda_{min}G(p,q,k)$ otherwises.

Proof Let $p = \frac{n-k}{2} - 1$, $q = \frac{n-k}{2} + 1$. We will show that

$\lambda_{\min}\big(G(p+1,q-1,k)\big) < \lambda_{\min}G(p,q,k)$.

By (5), we have

$$f(p,q,k;\lambda_{\min}G(p,q,k)) - f(p+1,q-1,k;\lambda_{\min}G(p,q,k)) = \lambda_{min}^2G(p,q,k) - 2k, \quad (9)$$

It implies that

$$\lambda_{min}^2G(p,q,k) = \frac{(pq+k)+\sqrt{(pq-k)^2+4kq}}{2}. \quad (10)$$

Thus

$$f(p+1,q-1,k;\lambda_{\min}G(p,q,k)) = -\left(\frac{(pq+k)+\sqrt{(pq-k)^2+4kq}}{2}\right) + 2k. \quad (11)$$

To make (11) < 0, we must show that

$$k^2 - (2n+10)k + (n^2+2n) > 0$$

or $k < n+5-\sqrt{8n+25}$. (12)

If $1 \le k \le n/2$ and $p = q-2$ then (12) holds except (n = 12, k = 6), (n = 8, k = 4) and (n = 7, k = 3). To complete the proof, we consider when (n = 12, k = 6), (n = 8, k = 4) and (n = 7, k = 3).

Case 1. n = 7, k = 3

Since p = q − 2 and n = 7, k = 3, $G(1,3,3)$ is only one graph in this case. By (5), $\lambda_{\min}G(1,3,3) = -\sqrt{6} = \lambda_{\min}G(2,2,3)$.

Case 2. *n* = 8*, k* = 4

Since p = q − 2 and n = 8, k = 3, $G(1,3,4)$, is only one graph in this case. By (5), $\lambda_{\min}G(1,3,4) = -\sqrt{7} = -2.645751311 < -2.613125930 = \lambda_{\min}G(2,2,4)$.

Case 3. *n* = 12*, k* = 6

Since p = q − 2 and n = 12, k = 6, $G(2,4,6)$ is only one graph in this case. By (5),

$$\lambda_{\min}G(2,4,6) = -2\sqrt{3} = \lambda_{\min}G(3,3,6).$$

This completes the proof. ❑

By Lemma 2.5, Lemma 2.6 and Lemma 2.7, we get the following result.

Theorem 2.8

Among all graphs $G(p,q,k)$ where $4 \le n \le 12$, $1 \le k \le \frac{n}{2}$, $n = p+q+k$, then minimizing graphs are

$$\begin{cases} G(2,4,6) \text{ and } G(3,3,6) & \text{if n = 12 and k = 6,} \\ G(1,3,4) & \text{if n = 8 and k = 4,} \\ G(1,3,3) \text{ and } G(2,2,3) & \text{if n = 7 and k = 3,} \\ G(\lfloor\frac{n-k}{2}\rfloor, \lceil\frac{n-k}{2}\rceil, k) & \text{otherwises.} \end{cases}$$

3. Characterization of minimizing graphs

In this section, we will characterize minimizing graphs in $\mathcal{G}_n^k$, where $4 \le n \le 12$, $1 \le k \le \frac{n}{2}$ and $n = p+q+k$.

Lemma 3.9 (Wang and Fan, 2012)

Let G be a minimizing graph in $\mathcal{G}_n^k$, and let x be a least vector of G. Then x contains no zero entries on vertices which are incident to cut edges of G.

Lemma 3.10 (Wang and Fan, 2012)

Let G be a minimizing graph in $\mathcal{G}_n^k$. Then all cut edges of G share a common vertex.

Remark 3.11(Wang and Fan, 2012)

Let $K_{1,k}$ be a star graph with vertex set $\{v_0, v_1, v_2, \dots, v_k\}$, where v_0 is the center of the star graph. For each $i \in \{1,2,3,\dots,k\}$, let $\mathcal{B}(a_i)$ is a connected graph of order a_i containing no cut edges. Let $\mathcal{B}(a_0; a_1, a_2, \dots, a_k)$ be the class of graphs obtained from $K_{1,k}$ by replacing each v_i by a graph $\mathcal{B}(a_i)$. Suppose that G is a minimizing graph in $\mathcal{G}_n^k$ with x as a least vector. Then all cut edges of G share a common vertex by the Lemma 3.10. Moreover, G is one graph of $\mathcal{B}(a_0; a_1, a_2, \dots, a_k)$ for some positive integers $a_0, a_1, a_2, \dots, a_k$.

Theorem 3.12

Let G be a minimizing graph in $\mathcal{G}_n^k$ where $4 \le n \le 12$ and $1 \le k \le \frac{n}{2}$. Then $G = G(p,q,k)$ for some positive integers p, q such that $p \ge 1, q \ge 1$ and $p+q+k = n$.

Proof Let G be a minimizing graph in $\mathcal{G}_n^k$. By Lemma 3.10 and Remark 3.11, we assume that G is one graph of $\mathcal{B}(a_0; a_1, a_2, \dots, a_k)$. where $\sum_{i=0}^{k} a_i = n$ and $a_i \ge 1$ for each $i \in \{0,1,2,\dots,k\}$.

Denote $G = B \in \mathcal{B}(a_0; a_1, a_2, \dots, a_k)$ Let x be a unit least vector of $\mathcal{B}$. Suppose that $u_0 \in V(\mathcal{B}_{a_0})$ joins u_i of $\mathcal{B}_{a_i}$ for each $i \in \{1,2,\dots,k\}$. We will show that $a_0 = n-k$ and $a_1 = a_2 = \cdots = a_k = 1$.

By contradiction, we assume that $a_i \ge 2$ for some $i = 1,2,\dots,k$ and replace all edges $u_i w$ by $u_0 w$, where $w \in \mathcal{N}_{\mathcal{B}_{a_i}}(u_i)$. We obtain a graph of $\widetilde{\mathcal{B}} \in \mathcal{B}(a_0 + a_i - 1; a_1, a_2, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_k) \subseteq \mathcal{G}_n^k$.

By Lemma 3.10, we have that all cut edges of $\widetilde{\mathcal{B}}$ are incident to u_0 and $|x(u_0)| = max_{v \in V(\widetilde{\mathcal{B}})}|x(v)|$.

By Lemma 2.3, we have $\lambda_{min}\widetilde{\mathcal{B}} \le \lambda_{min}\mathcal{B}$. If $\lambda_{min}\widetilde{\mathcal{B}} = \lambda_{min}\mathcal{B}$ then x is also a least vector of $\widetilde{\mathcal{B}}$ and $x(u_0) = x(u_i)$ and By Lemma 3.9, x be a unit least vector of $\widetilde{\mathcal{B}}$ then x contains no zero entries on vertices which are incident to cut edges of $\widetilde{\mathcal{B}}$.

From $\lambda x(u_i) = \sum_{w \in \mathcal{N}_{\mathcal{B}_{a_i}(u_i)}} x(w) = x(u_0)$, we have $\lambda_{min}\widetilde{\mathcal{B}} = 1$ and by Lemma 3.9, we get $x(u_i)$ and $x(u_0)$ are both nonzero, a contradiction. Then $\lambda_{min}\widetilde{\mathcal{B}} < \lambda_{min}\mathcal{B}$, a contradiction.

Next, we will show that $\mathcal{B}_{a_0}(a_0 = n-k)$ is a complete bipartite graph $K_{p,q}$ for some positive $p \ge 1, q \ge 1$. Let $V^+ = \{v \in V(\mathcal{B}_{a_0}) / x(v) \ge 0\}$ and $V^- = \{v \in V(\mathcal{B}_{a_0}) / x(v) < 0\}$. Since $\mathcal{B}_{a_0}$ contains $n-k$ $(\ge \frac{n}{2} > 1)$ vertices and (2), it contains vertices whose value has opposite sign to u_0. Then V^+ and V^- are both nonempty.

Case 1. $|V^+| \ge 2$ and $|V^-| \ge 2$

We will show that $\mathcal{B}_{a_0}(a_0 = n-k)$ is a complete bipartite graph. Suppose that $\mathcal{B}_{a_0}$ $(a_0 = n-k)$ is not a complete bipartite. Deleting the edges within V^+ or V^-, and adding all possible edges between V^+ and V^-, we get the graph $\tilde{G}$ in $\mathcal{G}_n^k$. By Equation (1), we have $x^TA(G)x = 2\sum_{uv \in E(G)} x(u)x(v) \ge 2\sum_{uv \in E(\tilde{G})} x(u)x(v) = x^TA(\tilde{G})x$. By equation (3), we have $\lambda_{\min}(\tilde{G}) \le x^TA(\tilde{G})x \le x^TA(G)x = \lambda_{\min}(G)$.

If $\lambda_{\min}(\tilde{G}) = \lambda_{\min}(G)$, then x is a least vector of $\widetilde{G}$ which implies that x contains no zero entries. Thus $x^TA(\tilde{G})x < x^TA(G)x$, a contradiction.

Case 2. $|V^+| = 1$ or $|V^-| = 1$

Without loss of generality, we may assume that $|V^-| = 1$. Deleting all edges within V^+ and adding all possible edges between V^+ and V^-, we get a tree T of order n. Thus $\lambda_{\min}(G) \ge \lambda_{\min}(T) \ge \lambda_{min}(K_{1,n-1})$, where $K_{1,n-1}$ is the unique minimizing graph among all trees graph of order n. Note that $K_{1,n-1} = G(1, n-k-1, k)$. We have

$$\lambda_{\min}(G) \ge \lambda_{\min}(T) \ge \lambda_{min}(K_{1,n-1}) = \lambda_{min}G(1, n-k-1, k) \qquad (13)$$

By Theorem 2.8, we divide our consideration into four cases.

Case 1. n = 7, k = 3

Then $|V^+| = 3$ and $|V^-| = 1$. It implies that

$\lambda_{\min}(G) \ge \lambda_{min}G(1, n-k-1, k) = \lambda_{min}G(1,3,3)$.

Since $(1,3,3) \in \mathcal{G}_n^k$, it contradics G is minimizing. Thus this case cannot occur.

Case 2. n = 8, k = 4

Then $|V^+| = 3$ and $|V^-| = 1$. It implies that

$\lambda_{\min}(G) \ge \lambda_{min}G(1, n-k-1, k) = \lambda_{min}G(1,3,4)$

Since $(1,3,4) \in \mathcal{G}_n^k$, it contradics G is minimizing. Thus this case cannot occur.

Case 3. n = 12, k = 6

By (12) and Theorem 2.8, we obtain

$$\lambda_{\min}(G) \ge \lambda_{min}G(1, n-k-1, k) = \lambda_{min}G(1,5,6) > \lambda_{min}G(2,4,6).$$

Since $G(2,4,6) \in \mathcal{G}_n^k$, a contradiction to G be minimizing. Thus this case cannot occur.

Case 4. the other case

By (12) and Theorem 2.8, we obtain

$$\lambda_{\min}(G) \geq \lambda_{min} G(1, n-k-1, k) > \lambda_{min} G(\lfloor \frac{n-k}{2} \rfloor, \lceil \frac{n-k}{2} \rceil, k).$$

As $n-k \geq \frac{n}{2} > 1$, $G\left(\lfloor \frac{n-k}{2} \rfloor, \lceil \frac{n-k}{2} \rceil, k\right) \in \mathcal{G}_n^k$,

Which yields a contradiction. Thus this case cannot occur. ❑

By Theorem 2.8 and Theorem 3.12, we obtain the main result;

Theorem 3.13

Among all graphs in $\mathcal{G}_n^k$ where $4 \leq n \leq 12$ and $1 \leq k \leq \frac{n}{2}$.

then minimizing graphs are
$$\begin{cases} G(2,4,6) \text{ and } G(3,3,6) & \text{if n = 12 and k = 6,} \\ G(1,3,4) & \text{if n = 8 and k = 4,} \\ G(1,3,3) \text{ and } G(2,2,3) & \text{if n = 7 and k = 3,} \\ G(\lfloor \frac{n-k}{2} \rfloor, \lceil \frac{n-k}{2} \rceil, k) & \text{otherwises.} \end{cases}$$

4. References

Bell, F.K., Cvetkovic, D., Rowlinson, P. and Simic, S.K., 2008, Graphs for which the least eigen-value is minimal I, Linear Algebra Appl. 429: 234-241.

Cvetkovic, D. and Rowlinson, P., 1990, The largest eigenvalue of a graph: A survey. Linear Multilinear Algebra 28: 3-33.

Fan, Y.Z., Wang, Y. and Gao, Y.B., 2008, Minimizing the least eigenvalues of unicyclic graphs with application to spectral spread, Linear Algebra Appl. 429: 577-588.

Powers, D.L., 1989, Bounds on graph eigenvalue, Linear Algebra Appl. 117: 1-6.

Wang, Y. and Fan, Y.Z., 2012, The least eigenvalue of graphs with cut edges. Graphs Combin. 28: 555-561.