บทความวิจัย (Research Article)

On Inverse Semipolygroups

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Received: June 10, 2018; Accepted: October 18, 2018

Abstract

In this paper, we introduce the notions of the inverse semipolygroup and the normal subsemipolygroup. Some properties of inverse semipolygroup and normal subsemipolygroup are described.

Keywords: semipolygroup; inverse semipolygroup; normal subsemipolygroup

1. Introduction

Marty (1934) first presented the concept of hyperstructure. It is well known that the class of hyperstructures is generalized from group theory. In a group, the combination between two elements (of a non-empty set) is an element but in a hypergroup, the combination between two elements is a set. The semihypergroups are an associative property of hypergroups which are based on the concept of hyperoperation. Later, Corsini (1993) introduced the fundamental theory of hyperstructures and many notions of hyperstructure can be found in his work. Next, the special subclasses of hypergroup called polygroups were studied by Comer (1996). He studied polygroups and applied hyperstructures with algebras and color schemes. Other researchers developed the concept of hyperstructure, for example, Davvaz (2000) proved new identities of strong regularity and fuzzy strong regularity on semihypergroups, and presented results on congruences of semihypergroups. Moreover, Davvaz (2010) considered the normal subpolygroups and homomorphisms between polygroups and identified the isomorphism theorems of polygroups. Furthermore, Jafarabadi *et al.* (2012) introduced new kinds of hyperstructure called simple and completely simple semihypergroups, and presented methods for constructing these new classes of hyperstructure and considered the regularity of semihypergroups and Davvaz (2013), later, discussed polygroup theory and related systems.

The purpose of this paper is to examine the notions of inverse semipolygroups and investigate their basic properties. Furthermore, we introduce the normal subsemipolygroup and some of their properties are also described.

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DOI: 10.14456/tjst.2019.40

2. Preliminary Results

This section gathers together results we shall use later. We describe the structure of the semipolygroup and then go on to introduce some important results related to it.

Definition 2.1 Let S be a non-empty set and let $P^*(S)$ be the set of all non-empty subsets of S. A *hyperoperation* on S is a map $\circ : S \times S \to P^*(S)$. A *polygroupoid* is a system $(S,\circ)$, where $\circ$ is a hyperoperation, i.e., $\emptyset \neq x \circ y = \circ(x,y) \subseteq S$ for all x,y of S. If A and B are non-empty subsets of S, then we denote

$$A \circ B = \bigcup_{a\in A,\ b\in B} a \circ b,$$

$x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. A polygroupoid $(S,\circ)$ is called a *semipolygroup* if, for all x,y,z of S, we have $(x\circ y)\circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v.$$

If a semipolygroup $(S,\circ)$ has the property that, $x \circ y = y \circ x$ for all x,y in S, we shall say that S is a *commutative semipolygroup*. If a semipolygroup $(S,\circ)$ contains an element e with the property that, for all x in S,

$$e \circ x = x \circ e = \{x\},$$

we say that e is an *identity element* of S, and that S is a *semipolygroup with identity*.

Example 2.2 Let $S = \{a,b,c\}$ with the following multiplication table:

$\cdot$	a	b	c
a	$\{a,b\}$	$\{a,b\}$	$\{c\}$
b	$\{a,b\}$	$\{a\}$	$\{c\}$
c	$\{c\}$	$\{c\}$	$\{a,b\}$

Then, $(S,\cdot)$ is a semipolygroup.

Example 2.3 Define a hyperoperation $\circ$ on $\mathbb{R}$ by $x \circ y = \{x,-x,y,-y\}$ for all x,y in $\mathbb{R}$. Then, $(\mathbb{R},\circ)$ is a semipolygroup.

Example 2.4 Let S be a non-empty set. Define a hyperoperation $\circ$ on S by $x \circ y = \{x,y\}$ for all x,y in S. Then, $(S,\circ)$ is a semipolygroup. For x,y in a semipolygroup $(S,\circ)$, we write the product of x,y as xy instead of $x \circ y$.

Remark: If a semipolygroup $(S,\circ)$ has no identity element, then we define a hyperoperation on S which has an adjoined element e as $e \circ x = x \circ e = \{x\}$ for all x in S, and $e \circ e = \{e\}$, and we can check that $S \cup \{e\}$ becomes a semipolygroup with identity. We define $S^1 = S$, if S has an identity elements, otherwise $S^1 = S \cup \{e\}$. We refer to S^1 as the *semipolygroup with identity obtained from S by adjoining an identity if necessary*. If a is an element of a semipolygroup S without identity, then Sa need not contain a. The following notations will be standard: $S^1a = Sa \cup \{a\}$ and $aS^1 = aS \cup \{a\}$.

Definition 2.5 Let a and b be elements of a semipolygroup S. An equivalence relation $\mathcal{L}$ on S is defined by the rule that $a\mathcal{L}b$ if and only if $S^1a = S^1b$. Similarly, we define the equivalence relation $\mathcal{R}$ on S by the rule that $a\mathcal{R}b$ if and only if $aS^1 = bS^1$.

Proposition 2.6 Let a and b be elements of a semipolygroup S. Then, $a\mathcal{L}b$ if and only if there exist x, $y \in S^1$ such that $xa \supseteq \{b\}$, $yb \supseteq \{a\}$ and $a\mathcal{R}b$ if and only if there exist u, $v \in S^1$ such that $au \supseteq \{b\}$, $bv \supseteq \{a\}$.

Proof: Let e be an identity element.

Suppose that $a\mathcal{L}b$. Then, $S^1a = S^1b$, i.e., $Sa \cup \{a\} = Sb \cup \{b\}$. If $a = b$, then there exist $x = e$, $y = e \in S^1$ such that $\{b\} = eb = xa$ and $\{a\} = ea = yb$. If $a \neq b$, then $b \in Sa$ and $a \in Sb$. That is, $\{b\} \subseteq xa$ for some $x \in S \subseteq S^1$ and $\{a\} \subseteq yb$ for some $y \in S \subseteq S^1$.

Conversely, assume that $\{b\} \subseteq xa$ and $\{a\} \subseteq yb$ for some x, $y \in S^1$. If $x = e$, then $\{b\} \subseteq ea = \{a\}$ implies $b = a$. Similarly, if $y = e$ we get $a = b$. Therefore, for $x = e$ or $y = e$, we have $S^1a = S^1b$.

Next, we assume that $x \neq e$ and $y \neq e$. If S has no identity element, then x, $y \in S$, and so $\{b\} \subseteq Sa$ and $\{a\} \subseteq Sb$. Now, let $p \in Sa \cup \{a\}$. If $p = a$, we get $p \in Sb \subseteq Sb \cup \{b\}$ because $\{a\} \subseteq Sb$. If $p \neq a$, then $p \in Sa$ and, for some $s \in S$, $p \in sa \subseteq s(Sb) \subseteq (sS)b \subseteq Sb \subseteq Sb \cup \{b\}$. Therefore, $Sa \cup \{a\} \subseteq Sb \cup \{b\}$. Next, we let $q \in Sb \cup \{b\}$. If $q = b$ we get $q \in Sa \subseteq Sa \cup \{a\}$ because $\{b\} \subseteq Sa$. If $q \neq b$, then $q \in Sb$ and, for some $s \in S$, $q \in sb \subseteq s(Sa) \subseteq (sS)a \subseteq Sa \subseteq Sa \cup \{a\}$, and so $Sb \cup \{b\} \subseteq Sa \cup \{a\}$. Hence, $Sa \cup \{a\} = Sb \cup \{b\}$. Similarly, if S has an identity element, say e, then $x, y \in S - \{e\}$ and $S^1 = S$. If $p' \in S^1a = Sa$, then there is $s_1 \in S$ such that $p' \in s_1a \subseteq s_1(yb) = (s_1y)b \subseteq Sb = S^1b$. If $q' \in S^1b = Sb$, then there is $s_2 \in S$ such that $q' \in s_2b \subseteq s_2(xa) \subseteq (s_2x)a \subseteq Sa = S^1a$. Hence, $S^1a = S^1b$ and thus, $a\mathcal{L}b$.

In the same way, we can prove that $a\mathcal{R}b$ if and only if there exist u, $v \in S^1$ such that $au \supseteq \{b\}$, $bv \supseteq \{a\}$. ∎

Definition 2.7 An element s in a semipolygroup S is called an *idempotent* if $s \in s^2$ and the element $a \in S$ is called a *scalar* if $|ax| = |xa| = 1$ for all x in S, where $|A|$ is the number of elements in a set A.

Definition 2.8 (Jafarabadi *et al.*, 2012) An element a of a semipolygroup S is called *regular* if there exists x in S such that $a \in axa$. The semipolygroup S is called *regular* if all elements are regular.

Definition 2.9 A non-empty subset T of a semipolygroup S is called a *subsemipolygroup* if $xy \subseteq T$ for all x, $y \in T$.

Proposition 2.10 (Jafarabadi *et al.*, 2012) If $(S,\cdot)$ and $(T,\circ)$ are semipolygroups, then the Cartesian product $S \times T$ becomes a semipolygroup if we define

$$(s,t) \diamond (s',t') = (s \cdot s') \times (t \circ t') = \bigcup_{x \in s\cdot s', y \in t\circ t'} \{(x,y)\}.$$

We refer to that semipolygroup as the direct product of S and T.

Definition 2.11 Let $(S,\circ)$ be a semipolygroup and ρ be an equivalence relation on S. Then, $A\bar{\rho}B$ means that for all $a \in A$ there exists $b \in B$ such that $a\rho b$ and for all $b' \in B$ there exists $a' \in A$ such that $a'\rho b'$; $A\bar{\bar{\rho}}B$ means that $a\rho b$ for all $a \in A$ and $b \in B$, where A and B are non-empty subsets of S. (We see that if $A\bar{\bar{\rho}}B$, then $A\bar{\rho}B$.)

(i) ρ is called *regular on the left* (*on the right*) if, for all x, y, a of $S, x\rho y \Rightarrow (ax)\bar{\rho}(ay)$ ($(xa)\bar{\rho}(ya)$, respectively);

(ii) ρ is called *strongly regular on the left* (*on the right*) if, for all x, y, a of $S, x\rho y \Rightarrow (ax)\bar{\bar{\rho}}(ay)$ ($(xa)\bar{\bar{\rho}}(ya)$, respectively);

(iii) ρ is called *regular* (*strongly regular*) if it is regular (strongly regular) on the left and on the right.

Remark: If ρ is a strongly regular relation, then ρ is a regular relation.

Theorem 2.12 (Davvaz, 2013) Let $(S,\circ)$ be a semipolygroup and ρ be an equivalence relation on S.

(i) If ρ is a regular relation on S, then the quotient set S/ρ is a semipolygroup with respect to the following the hyperoperation: $\rho(x)\otimes\rho(y) = \{\rho(z): z \in xy\}$.

(ii) If the hyperoperation defined by (i) is well-defined on S/ρ, then ρ is regular.

3. Inverse Semipolygroups

In this section, the concepts of inverse semipolygroups are introduced. Some results of inverse semipolygroup are proved.

Definition 3.1. If a is an element of a semipolygroup S, we say that a' is an *inverse* of a if $a \in aa'a$ and $a' \in a'aa'$.

Definition 3.2. A semipolygroup S will be called an *inverse semipolygroup* if there exists a unique unary operation $a \mapsto a^{-1}$ on S with the properties $(a^{-1})^{-1} = a$, $aa^{-1}a \ni a$,

$a \in bc$ impiles $b \in ac^{-1}$ and $c \in b^{-1}a$

for all a, b, c in S.

Notice that, if S is an inverse semipolygroup and $a \in S$, it follows that $a^{-1}aa^{-1} = a^{-1}(a^{-1})^{-1}a^{-1} \ni a^{-1}$, and thus, a^{-1} is an inverse of a.

Remark: Every inverse semipolygroup is regular.

Example 3.3. Let $S = \{a,b,c\}$ with the following multiplication table:

$\cdot$	a	b	c
a	$\{a,b,c\}$	$\{a,b\}$	$\{a,c\}$
b	$\{a,b\}$	$\{a,b,c\}$	$\{b,c\}$
c	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$

Then, $(S,\cdot)$ is an inverse semipolygroup.

Example 3.4. Define a hyperoperation ∘ on $\mathbb{R}^+$ by $x \circ y = \{xy\}$ for all x, $y \in \mathbb{R}^+$. Let x, y, $z \in \mathbb{R}^+$. Since $(x \circ y) \circ z = \{xy\} \circ z = \{(xy)z\} = \{x(yz)\} = x \circ \{yz\} = x \circ (y \circ z)$, $(\mathbb{R}^+,\circ)$ is a semipolygroup. Since $x \in \mathbb{R}^+$, there exist $x^{-1} = \frac{1}{x} \in \mathbb{R}^+$ such that $x \circ x^{-1} \circ x = x \circ \frac{1}{x} \circ x = \left\{x\frac{1}{x}\right\} \circ x = \{1\} \circ x = \{x\} \ni x$, and $(x^{-1})^{-1} = x$. Suppose that $w \in \mathbb{R}^+$ is an inverse of x. Then, $x \in x \circ w \circ x = \{xwx\}$, and hence, $w = \frac{1}{x}$. That is, $x^{-1} = \frac{1}{x}$ is unique. Suppose that $x \in y \circ z = \{yz\}$. Then, $x = yz$, so that $y = \frac{x}{z}$ and $z = \frac{x}{y}$. Thus, $y \in \{y\} = \left\{\frac{x}{z}\right\} = \left\{x\left(\frac{1}{z}\right)\right\} = x \circ \frac{1}{z} = x \circ z^{-1}$ and $z \in \{z\} = \left\{\frac{x}{y}\right\} = \left\{\left(\frac{1}{y}\right)x\right\} = \frac{1}{y} \circ x = y^{-1} \circ x$. Therefore, $(\mathbb{R}^+,\circ)$ is an inverse semipolygroup.

Proposition 3.5 Let a and b be elements of an inverse semipolygroup S. Then,

(i) $(ab)^{-1} = b^{-1}a^{-1}$;

(ii) if $a^{-1}a = b^{-1}b$ then $a\mathcal{L}b$;

(iii) if $aa^{-1} = bb^{-1}$ then $a\mathcal{R}b$.

Proof: Let a and b be elements of an inverse semipolygroup S. Then, there exist a^{-1}, $b^{-1} \in S$ such that $a \in aa^{-1}a$ and $b \in bb^{-1}b$.

(i) We have $ab \subseteq S$. Let $x^{-1} \in (ab)^{-1} = \{x^{-1}: x \in ab\}$. Then, $x \in ab$ implies $x^{-1} \in b^{-1}a^{-1}$. Hence, $(ab)^{-1} \subseteq b^{-1}a^{-1}$. Since a^{-1}, $b^{-1} \in S$, $b^{-1}a^{-1} \subseteq S$. Then, let $y \in b^{-1}a^{-1}$. This implies that $a^{-1} \in (b^{-1})^{-1}y = by$, and then $b \in a^{-1}y^{-1}$. It follows that $y^{-1} \in$

$(a^{-1})^{-1}b = ab$. Thus, since $(y^{-1})^{-1} \in (ab)^{-1}$, $y \in (ab)^{-1}$ and so $b^{-1}a^{-1} \subseteq (ab)^{-1}$. Consequently, $(ab)^{-1} = b^{-1}a^{-1}$.

(ii) Suppose that $a^{-1}a = b^{-1}b$. We have $a \in aa^{-1}a = ab^{-1}b = (ab^{-1})b$ and $b \in bb^{-1}b = ba^{-1}a = (ba^{-1})a$. Then, there exists $x \in ab^{-1} \subseteq S \subseteq S^1$, $y \in ba^{-1} \subseteq S \subseteq S^1$ such that $a \in xb$ and $b \in ya$. From Proposition 2.6, it follows that $a\mathcal{L}b$.

(iii) Similarly, $aa^{-1} = bb^{-1}$ implies $a\mathcal{R}b$.

■

Lemma 3.6 Let S be an inverse semipolygroup and $x \in S$. Every element in xx^{-1} and $x^{-1}x$ is regular.

Proof: Let S be an inverse semipolygroup and $x \in S$. Then, there exists a unique $x^{-1} \in S$ such that $xx^{-1} \subseteq S$. Let $a \in xx^{-1}$. This implies that $x^{-1} \in x^{-1}a$ and $x \in a(x^{-1})^{-1} = ax$. Thus, we have $a \in xx^{-1} \subseteq (ax)(x^{-1}a) = a(xx^{-1})a$. Thus, there exists $b \in xx^{-1}$ such that $a \in aba$ and hence, a is regular. Similarly, we also have that every element in $x^{-1}x$ is regular. ■

Lemma 3.7 Let S be an inverse semipolygroup. Then,

(i) if s is an idempotent of S, then $s^{-1} = s$;

(ii) if s is a scalar idempotent of S, then $ss^{-1} = \{s\}$.

Proof: Let S be an inverse semipolygroup.

(i) Let s be an idempotent of S. Then, there exists a unique $s^{-1} \in S$ such that $s \in ss^{-1}s$. Since $s \in s^2 = ss \subseteq sss$, s is an inverse of s. Thus, $s^{-1} = s$.

(ii) Let S be a scalar idempotent of S. By (i), $ss^{-1} = ss = \{s\}$. ■

Definition 3.8 A non-empty subset K of an inverse semipolygroup S is said to be an *inverse subsemipolygroup* of S if, under the hyperoperation in S, K itself forms an inverse semipolygroup.

Note that if K is an inverse subsemipolygroup of an inverse semipolygroup S, then we define K^{-1} to be the set $\{a^{-1}: a \in K\}$.

Lemma 3.9 A non-empty subset K of an inverse semipolygroup S is an inverse subsemipolygroup of S if and only if

(i) $a, b \in K$ implies $ab \subseteq K$ and

(ii) $a \in K$ implines $a^{-1} \in K$.

Proof: Let S be an inverse semipolygroup. First, suppose that K is an inverse subsemipolygroup of S and let a, $b \in K$. Since K is an inverse semipolygroup, $a^{-1} \in K$ and $ab \subseteq K$. Conversely, we assume that $a^{-1} \in K$ and $ab \subseteq K$ for all a, $b \in K$. Let a, b, $c \in K$. Since $K \subseteq S$, so a, b, $c \in S$. It implies that, since $ab \subseteq K$ and $(ab)c = a(bc)$, K is a semipolygroup. We also have that $a \in bc$ impiles $b \in ac^{-1}$ and $c \in b^{-1}a$. Since $a \in S$, there exists a unique $a^{-1} \in S$ such that $(a^{-1})^{-1} = a$ and $aa^{-1}a \ni a$. But $a^{-1} \in K$, so that K is an inverse semipolygroup. ■

Lemma 3.10 Let A and B be inverse subsemipolygroups of an inverse semipolygroup S. Then, for all x, y, a, $b \in S$,

(i) $x \in Ab$ implies $b \in A^{-1}x$ and $y \in aB$ implies $a \in yB^{-1}$ for all x, y, a, $b \in S$;

(ii) $(AB)^{-1} = B^{-1}A^{-1}$.

Proof: Let A and B be inverse subsemipolygroups of an inverse semipolygroup S.

(i) Let $x, y, a, b \in S$. Suppose that $x \in Ab$. Then, $x \in a_1 b$ for some $a_1 \in A$. It implies that $b \in {a_1}^{-1}x \subseteq A^{-1}x$. Similarly, we have that $y \in aB$ implies $a \in yB^{-1}$.

(ii) Let $y^{-1} \in (AB)^{-1}$. Then $y \in AB$ and thus, $y \in ab$ for some $a \in A$ and $b \in B$. Therefore, $y^{-1} \in (ab)^{-1} = b^{-1}a^{-1} \subseteq B^{-1}A^{-1}$. Now, let $x^{-1} \in B^{-1}A^{-1}$. Thus, there are $b^{-1} \in B^{-1}$ and $a^{-1} \in A^{-1}$ such that $x^{-1} \in b^{-1}a^{-1}$. It follows that $x = (x^{-1})^{-1} \in (b^{-1}a^{-1})^{-1} = (a^{-1})^{-1}(b^{-1})^{-1} = ab \subseteq AB$ and thus, $x^{-1} \in (AB)^{-1}$. Hence, $(AB)^{-1} = B^{-1}A^{-1}$. ■

Corollary 3.11 Let $a_1, a_2, a_3, \ldots, a_n$ be elements of an inverse semipolygroup S. Then, $(a_1a_2a_3 \cdots a_n)^{-1} = {a_n}^{-1}{a_{n-1}}^{-1}{a_{n-2}}^{-1} \cdots {a_1}^{-1}$ for every positive integer n.

Proof: Let $n \in \mathbb{N}$ and $a_1, a_2, a_3, \ldots, a_n$ be elements of an inverse semipolygroup S. Let $P(n)$ be the statement $(a_1a_2a_3 \cdots a_n)^{-1} = {a_n}^{-1}{a_{n-1}}^{-1}{a_{n-2}}^{-1} \cdots {a_1}^{-1}$. We see that $P(1)$ is true. Let $k \in \mathbb{N}$. Suppose that $P(k)$ is true. Then, $(a_1a_2a_3 \cdots a_k)^{-1} = {a_k}^{-1}{a_{k-1}}^{-1}{a_{k-2}}^{-1} \cdots {a_1}^{-1}$. Thus, using Lemma 3.1, ${a_{k+1}}^{-1}{a_k}^{-1}{a_{k-1}}^{-1} \cdots {a_1}^{-1} = {a_{k+1}}^{-1}({a_k}^{-1}{a_{k-1}}^{-1}{a_{k-2}}^{-1} \cdots {a_1}^{-1}) = {a_{k+1}}^{-1}(a_1a_2a_3 \cdots a_k)^{-1} = ((a_1a_2a_3 \cdots a_k)a_{k+1})^{-1} = (a_1a_2a_3 \cdots a_ka_{k+1})^{-1}$. Therefore, $P(k+1)$ is true. By the mathematical induction, we get $(a_1a_2a_3 \cdots a_n)^{-1} = {a_n}^{-1}{a_{n-1}}^{-1}{a_{n-2}}^{-1} \cdots {a_1}^{-1}$ for all $n \in \mathbb{N}$. ■

Lemma 3.12 Let S be an inverse semipolygroup and $a, b \in S$. The following statements (i) $aa^{-1} = ba^{-1}$; (ii) $aa^{-1} = ab^{-1}$; (iii) $a^{-1}a = b^{-1}a$; (iv) $a^{-1}a = a^{-1}b$; (v) $a \in ab^{-1}a$; (vi) $a \in aa^{-1}b$; are (i) ⟺ (ii) ⟹ (v), (iii) ⟺ (iv) ⟹ (vi) and (v) ⟺ (vi).

Proof: Let S be an inverse semipolygroup and $a, b \in S$. (1) $aa^{-1} = ba^{-1}$ iff $(aa^{-1})^{-1} = (ba^{-1})^{-1}$ iff $aa^{-1} = ab^{-1}$. Thus, (i) ⟺ (ii). (2) Suppose that $aa^{-1} = ba^{-1}$. Since $a^{-1} \in S$, and $a^{-1} \in a^{-1}aa^{-1} = a^{-1}ba^{-1}$, so $a = (a^{-1})^{-1} \in (a^{-1}ba^{-1})^{-1} = ab^{-1}a$. Thus, (i) ⟹ (v). (3) $a^{-1}a = a^{-1}b$ iff $(a^{-1}a)^{-1} = (a^{-1}b)^{-1}$ iff $a^{-1}a = b^{-1}a$. Thus, (iii) ⟺ (iv). (4) Suppose that $a^{-1}a = b^{-1}a$. Since $a \in S$, so $a^{-1} \in a^{-1}aa^{-1} = b^{-1}aa^{-1}$. It implies that $a \in aa^{-1}b$. Thus, (iii) ⟹ (vi). (5) Suppose that $a \in aa^{-1}b$. Then $a^{-1} \in b^{-1}aa^{-1}$. By Lemma 3.10, we have $a^{-1} \in (b^{-1}a)^{-1}a^{-1} = a^{-1}ba^{-1}$ and thus, $a \in ab^{-1}a$.

Conversely, let $a \in ab^{-1}a$. Then, $a^{-1} \in a^{-1}ba^{-1}$. By Lemma 3.10, we have $a^{-1} \in (a^{-1}b)^{-1}a^{-1} = b^{-1}aa^{-1}$ and thus, $a \in aa^{-1}b$. Therefore, (v) ⟺ (vi). ■

Theorem 3.13 Let $(S,\cdot)$ and $(T,\circ)$ be two inverse semipolygroups. Then, the product $S \times T$, with respect to hyperoperation defined by Proposition 2.10, is an inverse semipolygroup, where $(s,t)^{-1} = (s^{-1}, t^{-1})$ for all $(s,t) \in S \times T$.

Proof: By Proposition 2.10, $S \times T$ is a semipolygroup. Let $(s,t), (s_1,t_1), (s_2,t_2), (s_3,t_3) \in S \times T$. We see that $((s,t)^{-1})^{-1} = (s,t)$. Now, $(s,t) \diamond (s,t)^{-1} \diamond (s,t) = \bigcup_{x \in s \cdot s^{-1} \cdot s, y \in t \circ t^{-1} \circ t}\{(x,y)\}$.

Since $s \in s \cdot s^{-1} \cdot s$ and $\in t \circ t^{-1} \circ t$, $(s,t) \in (s,t) \diamond (s,t)^{-1} \diamond (s,t)$. Suppose now that $(s_1,t_1) \in (s_2,t_2) \diamond (s_3,t_3) = \bigcup_{x \in s_2 \cdot s_3, y \in t_2 \circ t_3}\{(x,y)\}$. Thus, $s_1 \in s_2 \cdot s_3$ and $t_1 \in t_2 \circ t_3$. Therefore, $s_2 \in s_1 \cdot {s_3}^{-1}$, $s_3 \in {s_2}^{-1} \cdot s_1$, $t_2 \in t_1 \circ {t_3}^{-1}$ and $t_3 \in {t_2}^{-1} \circ t_1$. Hence, $(s_1,t_1) \diamond (s_3,t_3)^{-1} = \bigcup_{p \in s_1 \cdot {s_3}^{-1}, q \in t_1 \circ {t_3}^{-1}}\{(p,q)\} \ni (s_2,t_2)$ and $(s_2,t_2)^{-1} \diamond (s_1,t_1) = \bigcup_{u \in {s_2}^{-1} \cdot s_1, v \in {t_2}^{-1} \circ t_1}\{(u,v)\} \ni (s_3,t_3)$. ■

4. Normal Subsemipolygroups

In this section, the notions of normal subsemipolygroup are presented and some results of normal subsemipolygroup are established.

Definition 4.1 A non-empty subset N of an inverse semipolygroup S is a *normal subsemipolygroup* of S if

(i) N is an inverse subsemipolygroup;

(ii) if S is idempotent of S, then $s \in N$; and

(iii) if $a \in N$, then $x^{-1}ax \subseteq N$ for all $x \in S$.

Example 4.2 Define a hyperoperation $\circ$ on $\mathbb{Q}^+$ by $x \circ y = \{xy\}$ for all x, $y \in \mathbb{Q}^+$. By Example 3.4, we have $(\mathbb{R}^+,\circ)$ is an inverse semipolygroup. First, we see that $\mathbb{Q}^+ \subseteq \mathbb{R}^+$. Let $x, y, z \in \mathbb{Q}^+$. Then, $x = \frac{m}{n}$, $y = \frac{p}{q}$ and $z = \frac{u}{v}$ where $m, n, p, q, u, v \in \mathbb{Z}^+$. Thus, we have $x \circ y = \left\{\frac{mp}{nq}\right\} \subseteq \mathbb{Q}^+$. Since $x = \frac{m}{n} \in \mathbb{Q}^+$, there exists $x^{-1} = \frac{n}{m} \in \mathbb{Q}^+$ such that $x \circ x^{-1} \circ x \ni x$ and $(x^{-1})^{-1} = x$. It is clear that x^{-1} is unique. Suppose that $x \in y \circ z$. Then, $x = \frac{pu}{qv}$ and it implies that $y = \frac{mv}{nu}$ and $z = \frac{qm}{pn}$. Thus, $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$. Therefore, $(\mathbb{Q}^+,\circ)$ is an inverse semipolygroup.

From Example 3.4, we have $1 \in 1 \circ 1 = \{1\}$. Suppose that a is an idempotent of $\mathbb{R}^+$. This means that $a \in a \circ a$. Then $a = a^2$ and thus, $a = 1$. Consequently, there is only 1 is an idempotent in $\mathbb{R}^+$ and it is obvious that 1 is an idempotent in $\mathbb{Q}^+$. Let $r \in \mathbb{R}^+$. Then, there exists a unique $r^{-1} = \frac{1}{r} \in \mathbb{R}^+$ such that $r^{-1} \circ x \circ r = \{x\} \subseteq \mathbb{Q}^+$. Therefore, $\mathbb{Q}^+$ is a normal subsemipolygroup of an inverse semipolygroup $\mathbb{R}^+$.

Corollary 4.3 Let N be a normal subsemipolygroup of an inverse semipolygroup S and let $a \in S$. Then, $Na = Nb$ for all $b \in Na$.

Proof: Let $a \in S$ and let $b \in Na$. Then, $b \in na$ for some $n \in N$. If $x \in Na$, then $x \in n_1 a$ for some $n_1 \in N$. Since $b \in na$, $a \in n^{-1}b$. Therefore, $x \in n_1 a \subseteq n_1(n^{-1}b) = (n_1 n^{-1})b \subseteq Nb$. Similarly, we let $y \in Nb$. Thus, $y \in n_2 b$ for some $n_2 \in N$. It follows that $y \in n_2 b \subseteq n_2(na) = (n_2 n)a \subseteq Na$, and thus, $Na = Nb$. ■

Corollary 4.4 Let K and N be inverse subsemipolygroups of an inverse semipolygroup S with N normal in S. Then, $N \cap K$ is a normal subsemipolygroup of K if $N \cap K$ is a non-empty set.

Proof: Suppose that $N \cap K$ is a non-empty set. Let $a, b \in N \cap K$. It follows that $ab \subseteq N$ and $ab \subseteq K$, that is, $ab \subseteq N \cap K$. Now, we get that $N \cap K$ is a subsemipolygroup of K. Since $a \in N$ and $a \in K$, we have that $a^{-1} \in N$, $a^{-1} \in K$. Then, $a^{-1} \in N \cap K$. Thus, $N \cap K$ is an inverse semipolygroup. Suppose now that S is an idempotent of K. Because N is normal in

S and $s \in K \subseteq S$, it implies that $s \in N$. Then, $s \in N \cap K$.

Suppose that $a \in N \cap K$ and $k \in K$. Since N is normal, $k^{-1}ak \subseteq N$. Since a, k, $k^{-1} \in K$, we also get that $k^{-1}ak \subseteq K$. Therefore, $k^{-1}ak \subseteq N \cap K$. Hence, $N \cap K$ is a normal subsemipolygroup of K as required. ■

Definition 4.5 Let N be a normal subsemipolygroup of an inverse semipolygroup S. Define the relation τ on S by $(x, y) \in \tau$ (or $x \tau y$) if and only if $xy^{-1} \cap N \neq \emptyset$.

Lemma 4.6 The relation τ is an equivalence relation on an inverse semipolygroup S.

Proof: Suppose that $x \in S$. Let $n \in N$. We have $xn \subseteq S$. Then, there exists $a \in xn$ such that $x \in an^{-1}$ and $x^{-1} \in na^{-1}$. This implies that $xx^{-1} \subseteq (an^{-1})(na^{-1}) \subseteq aNa^{-1} \subseteq N$. Hence, $xx^{-1} \cap N \neq \emptyset$. Therefore, $x \tau x$, and thus, τ is reflexive.

Suppose that $x \tau y$, where x, $y \in S$. That is, there exists $a \in xy^{-1} \cap N$, i.e., $a \in xy^{-1}$ and $a \in N$. It follows that $a^{-1} \in yx^{-1}$. Since $a \in N$, $a^{-1} \in N$. Hence, $a^{-1} \in yx^{-1} \cap N$, that is, $yx^{-1} \cap N \neq \emptyset$ or $y \tau x$, and thus, τ is symmetric.

Assume that $x \tau y$ and $y \tau z$, where $x, y, z \in S$. Then, there are $a \in xy^{-1} \cap N$ and $b \in yz^{-1} \cap N$, that is, $a \in xy^{-1}$, $a \in N$ and $b \in yz^{-1}$, $b \in N$. Thus, we get $x \in ay$ and $z^{-1} \in y^{-1}b$. It follows that $z^{-1}x \subseteq (y^{-1}b)(ay) = y^{-1}(ba)y \subseteq N$. We see that $x^{-1}z = (z^{-1}x)^{-1} \subseteq N$. Let $v \in x^{-1}z$. Then, $z \in (x^{-1})^{-1}v = xv$ and thus, $x \in zv^{-1}$. Therefore, $xz^{-1} \subseteq (zv^{-1})z^{-1} \subseteq N$, we deduce that $xz^{-1} \cap N \neq \emptyset$ or $x \tau z$. Hence, τ is transitive, as required. ■

Lemma 4.7 The equivalence relation τ on an inverse semipolygroup S is a strongly regular relation.

Proof: Let $x, y, a \in S$. Suppose that $x \tau y$. Then, $xy^{-1} \cap N \neq \emptyset$. Let $u \in xa$ and $v \in ya$. It follows that $v^{-1} \in a^{-1}y^{-1}$. We now have $v^{-1}u \subseteq (a^{-1}y^{-1})(xa) = a^{-1}(y^{-1}x)a$. Since $xy^{-1} \cap N \neq \emptyset$, there exists $k \in xy^{-1}$ and $k \in N$ such that $x \in ky$. Now, we get that $y^{-1}x \subseteq y^{-1}ky \subseteq N$, because N is normal. It is obvious that $v^{-1}u \subseteq N$. That is $v^{-1}u \cap N \neq \emptyset$. Then, there exists $w \in v^{-1}u \cap N$ such that $v^{-1} \in wu^{-1}$. Thus, $uv^{-1} \subseteq uwu^{-1} \subseteq N$. That is, $uv^{-1} \cap N \neq \emptyset$ and thus, $u \tau v$. Therefore, τ is strongly regular on the right.

Let $m \in ax$ and $n \in ay$. Then, $m^{-1} \in x^{-1}a^{-1}$ and $n^{-1} \in y^{-1}a^{-1}$. It follows that $m^{-1} \tau n^{-1}$ because τ is strongly regular on the right. Thus, $m^{-1}n \cap N \neq \emptyset$ and there exists $p \in m^{-1}n \cap N$. Then, $m^{-1} \in pn^{-1}$. Thus, $nm^{-1} \subseteq npn^{-1} \subseteq N$. That is, $nm^{-1} \cap N \neq \emptyset$ or $n \tau m$. Therefore, τ is strongly regular on the left. Hence, τ is a strongly regular relation. ■

Let $\tau(x)$ be the equivalence class of the element x of an inverse semipolygroup S and let S/τ be the quotient set: $S/\tau = \{\tau(x): x \in S\}$. We defined the hyperoperation $\otimes$ on S/τ as follows: $\tau(x) \otimes \tau(y) = \{\tau(z): z \in xy\}$.

Lemma 4.8 Let N be a normal subsemipolygroup of an inverse semipolygroup S and let τ be an equivalence relation on S. Then, $\tau(x) = Nx$ for all $x \in S$.

Proof: Suppose that $y \in Nx$. Thus, there exists $n \in N$ such that $y \in nx$, which implies that $n \in yx^{-1}$. Hence, $yx^{-1} \cap N \neq \emptyset$. That is, $y\,\tau\,x$ and thus, $y \in \tau(x)$. Now, we have $Nx \subseteq \tau(x)$. Next, we let $y \in \tau(x)$. Then, $x\,\tau\,y$ or $xy^{-1} \cap N \neq \emptyset$. Hence, there exists $a \in xy^{-1} \cap N$ such that $y^{-1} \in x^{-1}a$. Thus, $y \in a^{-1}x \subseteq Nx$. Therefore, $\tau(x) \subseteq Nx$, and thus $\tau(x) = Nx$. ■

Lemma 4.9 The quotient set S/τ is an inverse semipolygroup with respect to the hyperoperation $\otimes$.

Proof: By Theorem 2.12, S/τ is a semipolygroup. For all $\tau(x) \in S/\tau$, since $(\tau(x))^{-1} = \{y^{-1} : y \in \tau(x)\}$, $\left(\left(\tau(x)\right)^{-1}\right)^{-1} = \{(y^{-1})^{-1} : y^{-1} \in \left(\tau(x)\right)^{-1}\} = \{y : y \in \tau(x)\} = \tau(x)$. Next, we show that $\tau(x) \in \tau(x)\otimes\left(\tau(x)\right)^{-1}\otimes\tau(x)$. We now have $(\tau(x)\otimes\left(\tau(x)\right)^{-1})\otimes\tau(x) = \{\tau(m) : m \in xx^{-1}\}\otimes\tau(x) = \bigcup_{m\in xx^{-1}}\{\tau(n) : n \in mx\}$. Since $x \in xx^{-1}x = (xx^{-1})x$, there exists $n \in xx^{-1}$ such that $x \in nx$. Hence,

$\tau(x) \in \bigcup_{m\in xx^{-1}}\{\tau(n) : n \in mx\} = \tau(x)\otimes\left(\tau(x)\right)^{-1}\otimes\tau(x)$. Let $\tau(x)$, $\tau(y)$, $\tau(z) \in S/\tau$. Suppose that $\tau(x) \in \tau(y)\otimes\tau(z)$. Then, $x \in yz$. Thus, $y \in xz^{-1}$ and $z \in y^{-1}x$. We then have that $\tau(y) \in \tau(x)\otimes(\tau(z))^{-1}$ and $\tau(z) \in \left(\tau(y)\right)^{-1}\otimes\tau(x)$. Hence, S/τ is an inverse semipolygroup. ■

Lemma 4.10 Let S be an inverse semipolygroup and N be a normal subsemipolygroup of S. Then, Ker $\tau = \bigcup_{e\in E}\tau(e)$ is a normal subsemipolygroup of S, where $E = \{f \in S : f$ is an idempotent of $S\}$.

Proof: Let $x, y \in$ Ker τ. Then, there exist $e, f \in E$ such that $x \in \tau(e)$ and $y \in \tau(f)$. So, $ex^{-1} \cap N \neq \emptyset$ and $fy^{-1} \cap N \neq \emptyset$. Thus, there exist $m \in ex^{-1} \cap N$ and $n \in fy^{-1} \cap N \neq \emptyset$. Thus, $x^{-1} \in e^{-1}m = em$ and $y^{-1} \in f^{-1}n = fn$. Let $a \in xy$. Then $a^{-1} \in y^{-1}x^{-1} \subseteq (fn)(em) \subseteq N$. Hence, $ea^{-1} \subseteq N$ and thus, $ea^{-1} \cap N \neq \emptyset$. We get that $a \in \tau(e)$, that is $a \in$ Ker τ and thus, $xy \subseteq$ Ker τ.

Next, suppose that $k \in$ Ker τ. Then, we have $ek^{-1} \cap N \neq \emptyset$ for some $e \in E$. Thus, there exists $u \in ek^{-1} \cap N$ and this implies that $k^{-1} \in eu$. Then $k \in u^{-1}e$. It follows that $ek \subseteq N$. Thus, $e(k^{-1})^{-1} \cap N \neq \emptyset$. That is, $k^{-1} \in \tau(e)$. Certainly, $k^{-1} \in$ Ker τ, and thus, Ker τ is an inverse subsemipolygroup.

Let $p \in S$ and let $q \in p^{-1}kp$. Then, $q^{-1} \in p^{-1}k^{-1}p$. Since $u \in ek^{-1}$, $k^{-1} \in eu$. We deduce that $q^{-1} \in p^{-1}k^{-1}p \subseteq p^{-1}Np \subseteq N$ and it also follows that $eq^{-1} \subseteq N$. Thus, $eq^{-1} \cap N \neq \emptyset$ and $q \in \tau(e)$. Therefore, $q \in$ Ker τ and thus, $p^{-1}kp \in$ Ker τ. Then, we have that Ker τ is a normal subsemipolygroup of S. ■

5. Acknowledgements

Appreciation is extended to the Department of Mathematics and Statistics, Thammasat University for the support that made this research project possible.

6. References

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