Functional Equations Characterizing the Tangent Function Over a Convex Polygon II

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Abstract

The functional equation $f(x)+f(y)+f(z)=f(x)f(y)f(z)$, satisfied by the three angles x, y and z of a non-degenerate triangle, was shown to characterize the tangent function by Benz in 2004. This result has been generalized by the authors to a functional equation, with n parameters representing the angles of a non-degenerate convex n-gon. Here, it is shown that there are other different but similar functional equations characterizing the tangent function.

Keywords: functional equation; tangent function; convex polygon

1. Introduction

Trigonometric functions satisfy a number of identities especially when combined with the angles of a triangle; some of best known are (Hall *et al*., 1957): If $A+B+C=\pi$, then

$$(1.1)\quad \begin{cases} \sin A+\sin B+\sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}; \\ \cos A+\cos B+\cos C = 1+4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}; \\ \tan A+\tan B+\tan C = \tan A\tan B\tan C. \end{cases}$$

among these three identities, (1.1) is simplest and most appealing because it involves only one function. Motivated by (1.1), Davison (2003) showed that the functional equation

$$(1.2)\quad f(x)+f(y)+f(z)=f(x)f(y)f(z),$$

under some conditions, is equivalent to the functional equation $g(x)g(y)g(z)=1$. Benz (2004), confirming this result of Davison (2003), proved that the general solution of

$$(1.3)\quad f(x)+f(y)+f(z)=f(x)f(y)f(z),$$

with x, y and z being the three angles of a non-degenerate triangle, is the tangent function. Hengkrawit *et al.* (2014) extended this result by showing that the functional equation

$$(1.4)\quad \sum_{i=1}^{n} f(x_i) = \sum_{M=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^{M+1} \sum_{1\le i_1<\cdots<i_{2M+1}\le n} f(x_{i_1})\cdots f(x_{i_{2M+1}}),$$

characterizes the tangent function, where $x_1, x_2, ..., x_n \ (n \ge 3)$ represent the angles of a

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non-degenerate convex n-gon. There then arises a natural question whether there are other similar yet different functional equations that can be used to characterize the tangent function. This paper answers this question through the following theorem.

Theorem 1.1 Let n be an odd positive integer ≥ 3. The functions $f: I \to \mathbb{R} \setminus \{0\}$, $I = (0, \pi)$ satisfying

(1.5) $\sum_{M=1}^{\frac{n-1}{2}} (-1)^{M+1} \sum_{1 \le i_1 < \cdots < i_{2M} \le n} f\left(\frac{x_{i_1}}{2}\right) \cdots f\left(\frac{x_{i_{2M}}}{2}\right) = 1,$

$x_i \in I \ (i = 1, \ldots, n),$ subject to the two conditions

(1.6) $x_1 + \cdots + x_n = (n-2)\pi,$

(1.7) $1 + \sum_{M=1}^{\frac{n-1}{2}} (-1)^M \sum_{1 \le i_1 < \cdots < i_{2M} \le n-1} f(\frac{x_{i_1}}{2}) \cdots f(\frac{x_{i_{2M}}}{2}) \neq 0,$

are given by $f(x) = \tan\left(k\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{s\pi}{2n}\right)$ $(s = 1, 3, \ldots, n-2),$ where k is a fixed constant belonging to the range $\max\left\{-\frac{s}{2}, \frac{s-n}{n-2}\right\} < k < \min\left\{\frac{s}{n-2}, \frac{n-s}{2}\right\}.$

Theorem 1.2 Let n be an even positive integer ≥ 4. The functions $f: I \to \mathbb{R} \setminus \{0\}$, $I = (0, \pi)$ satisfying

(1.8) $\sum_{M=0}^{\frac{n-2}{2}} (-1)^M \sum_{1 \le i_1 < \cdots < i_{2M+1} \le n} f\left(\frac{x_{i_1}}{2}\right) \cdots f\left(\frac{x_{i_{2M+1}}}{2}\right) = 0,$

$x_i \in I \ (i = 1, \ldots, n),$ subject to the two conditions

(1.9) $x_1 + \cdots + x_n = (n-2)\pi,$

(1.10) $1 + \sum_{M=1}^{\frac{n-2}{2}} (-1)^M \sum_{1 \le i_1 < \cdots < i_{2M} \le n-1} f\left(\frac{x_{i_1}}{2}\right) \cdots f\left(\frac{x_{i_{2M}}}{2}\right) \neq 0,$

are given by $f(x) = \tan\left(k\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{\ell\pi}{n}\right)$ $(\ell = 1, 2, \ldots, n-1),$ where k is a fixed constant belonging to the range $\max\left\{-\ell, \frac{2\ell-n}{n-2}\right\} < k < \min\left\{\frac{2\ell}{n-2}, \frac{n-2\ell}{2}\right\}.$

The result of Theorem 1.1 is a generalization of the identity $\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} + \tan\frac{A}{2}\tan\frac{B}{2} = 1,$ while that of Theorem 1.2 is a generalization of a similar identity for a quadrangle.

2. Two Lemmas

We first prove two lemmas. The first lemma shows that a function with constant sum over the angles of a convex polygon must be a linear function.

Lemma 2.1 Let $n \in \mathbb{N}$, $n \geq 3$, $I_1 = (0, \pi/2)$. If the functions $\phi: I_1 \to I_1$ satisfy

(2.1) $\sum_{i=1}^{n} \phi(x_i) = \frac{s\pi}{2}$ if n is odd

(2.2) $\sum_{i=1}^{n} \phi(x_i) = \ell\pi$ if n is even, for some fixed $s \in \{1, 3, 5 \ldots, (n-2)\}$, $\ell \in \{1, 2, 3, \ldots, (n-1)\}$ and $x_i \in I_1$ $(i = 1, \ldots, n)$ satisfying

(2.3) $\sum_{i=1}^{n} x_i = \frac{(n-2)\pi}{2}$, then $\phi(x) = k_1\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{s\pi}{2n}$ if n is odd, and $\phi(x) = k_2\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{\ell\pi}{n}$ if n is even, where k_1 and k_2 are fixed constants belonging to the range $\max\left\{-\frac{s}{2}, \frac{s-n}{n-2}\right\} < k_1 < \min\left\{\frac{s}{n-2}, \frac{n-s}{2}\right\}$ and $\max\left\{-\ell, \frac{2\ell-n}{n-2}\right\} < k_2 < \min\left\{\frac{2\ell}{n-2}, \frac{n-2\ell}{2}\right\}.$

Proof Let $J = \left(-(n-2)\pi/2n, \pi/n\right)$. Define

$\psi : J \to I_1$ by $\psi(x) = \phi\left(x + \dfrac{(n-2)\pi}{2n} \right)$ $(x \in J)$. Observe that if $x \in J$, then $x + \left((n-2)\pi / 2n\right) \in I_1$.

● If n is odd, then from (2.1) and (2.3) we get

(2.4) $\sum_{i=1}^{n} \psi(x_i) = \dfrac{s\pi}{2}$ $(x_i \in J)$ subject to $\sum_{i=1}^{n} x_i = 0$. Putting $x_i = 0$ $(i = 1, \dots, n)$ in (2.4), we have

(2.5) $\psi(0) = \dfrac{s\pi}{2n}$. Let $H = \left(-\pi / n, \pi / n\right)$, we see that $x \in H$, and so $-x \in H$. Thus, (2.4) gives $\sum_{i=1}^{n-2} \psi(0) + \psi(x) + \psi(-x) = \dfrac{s\pi}{2}$ $(x \in H)$. Combining this last relation with (2.5), we get

(2.6) $\psi(-x) = \dfrac{s\pi}{n} - \psi(x)$ $(x \in H)$. Next, let $x, y \in H$ be such that $x + y \in H$. Thus, (2.4) gives $\sum_{i=1}^{n-3} \psi(0) + \psi(x) + \psi(y) + \psi(-(x+y)) = \dfrac{s\pi}{2}$ $(x, y, x+y \in H)$. Combining this with (2.5) and (2.6), we have

(2.7) $\psi(x+y) = \psi(x) + \psi(y) - \dfrac{s\pi}{2n}$ $(x, y, x+y \in H)$. Taking $x \in \left(-(n-2)\pi / 2n, -\pi / n\right]$, $y \in \left[0, \pi / n\right)$ with $x + y \in \left(-\pi / n, 0\right)$, since $-(x+y) \in \left(0, \pi / n\right)$, the relation (2.4) gives $\sum_{i=1}^{n-3} \psi(0) + \psi(x) + \psi(y) + \psi(-(x+y)) = \dfrac{s\pi}{2}$. .Using (2.5) and (2.6), we have

(2.8) $\psi(x) = \psi(x+y) - \psi(y) + \dfrac{s\pi}{2n}$, for $x \in \left(-(n-2)\pi / 2n, -\pi / n\right]$, $y \in \left[0, \pi / n\right)$ with $x + y \in \left(-\pi / n, 0\right)$. The relations (2.5), (2.6), (2.7) and (2.8) suggest that the function ψ can be transformed into an additive function. To verify this, define $\beta : J \to \left(-s\pi / 2n, (n-s)\pi / 2n\right)$ by

(2.9) $\beta(x) = \psi(x) - \dfrac{s\pi}{2n}$ $(x \in J)$. From (2.5) and (2.9), we get

(2.10) $\beta(0) = 0$. From (2.6) and (2.9), we get

(2.11) $\beta(-x) = -\beta(x)$ $(x \in H)$. From (2.7) and (2.9), we get

(2.12) $\beta(x+y) = \beta(x) + \beta(y)$ $(x, y, x+y \in H)$. By Remark 1.73 of (Kannappan, 2009, p. 57), there exists a unique additive function $A : \mathbb{R} \to \mathbb{R}$ satisfying (2.12) over $\mathbb{R}$, which is an extension of β, viz, $A|_H = \beta$. Since A is bounded on H, by (Aczel *et al.*, 1989, Corollary 5 on p. 15), we have $A(x) = k_1 x$ $(x \in \mathbb{R})$, for some constant k_1, and consequently,

(2.13) $\beta(x) = k_1 x$ $(x \in H)$. From (2.8), (2.9) and (2.13), for $x \in \left(-(n-2)\pi / 2n, -\pi / n\right]$, $y \in \left[0, \pi / n\right)$ with $x + y \in \left(-\pi / n, 0\right)$, we get

(2.14) $\beta(x) = \beta(x+y) - \beta(y) = k_1(x+y) - k_1 y = k_1 x$, which yields $\beta(x) = k_1 x$ $(x \in J)$. Since β is the map from J into $\left(-s\pi / 2n, (n-s)\pi / 2n\right)$, we have $\max\left\{-\dfrac{s}{2}, \dfrac{s-n}{n-2}\right\} < k_1 < \min\left\{\dfrac{s}{n-2}, \dfrac{n-s}{2}\right\}$. By the definition of β, we have $\psi(x) = k_1 x + \dfrac{s\pi}{2n}$ $(x \in J)$. By the definition of ψ, we have $\phi(x) = k_1\left(x - \dfrac{(n-2)\pi}{2n} \right) + \dfrac{s\pi}{2n}$ $(x \in I_1)$.

● If n is even, the desired result follows by a similar proof and is omitted. □

The second lemma is an identity for the expansion of the tangent function over a convex polygon.

Lemma 2.2 (Hengkrawit *et al.*, 2014, Lemma 2.1) Let $n \in \mathbb{N},\ n \ge 3$, let $A_1, A_2, \ldots, A_{n-1} \in (0, \pi)$, and let $\sigma_1(n) = \sum_{M=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} (-1)^M \sum_{1 \le i_1 < i_2 < \cdots < i_{2M} \le n-1} \tan A_{i_1} \tan A_{i_2} \cdots \tan A_{i_{2M}}$,

$\sigma_2(n) = \sum_{M=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} (-1)^M \sum_{1 \le i_1 < i_2 < \cdots < i_{2M+1} \le n-1} \tan A_{i_1} \tan A_{i_2} \cdots \tan A_{i_{2M+1}}$.

If $1 + \sigma_1(n) \ne 0$, then $\tan(A_1 + \cdots + A_{n-1}) = \dfrac{\sigma_2(n)}{1 + \sigma_1(n)}$.

3. Proof of Theorem 1.1

Let $f : I \to \mathbb{R} \setminus \{0\}$ satisfy (1.5) subject to the conditions (1.6) and (1.7). For a suitable bijection (to be determined) $\phi : I \to I$, put

(3.1) $f(x) = \tan\big(\phi(x)\big)\ (x \in I),\ \phi : I \to I.$

From (1.5), we have

$$\begin{aligned}
1 &= \sum_{M=1}^{\frac{n-1}{2}} (-1)^{M+1} \sum_{1 \le i_1 < \cdots < i_{2M} \le n} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right) \cdots \tan\left(\phi\left(\frac{x_{i_{2M}}}{2}\right)\right) \\
&= \sum_{M=1}^{\frac{n-1}{2}} (-1)^{M+1} \sum_{1 \le i_1 < \cdots < i_{2M} \le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right) \cdots \tan\left(\phi\left(\frac{x_{i_{2M}}}{2}\right)\right) \\
&\quad + \left[\sum_{M=1}^{\frac{n-1}{2}} (-1)^{M+1} \sum_{1 \le i_1 < \cdots < i_{2M-1} \le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right) \cdots \tan\left(\phi\left(\frac{x_{i_{2M-1}}}{2}\right)\right)\right] \tan\left(\phi\left(\frac{x_n}{2}\right)\right) \\
&= \sum_{M=1}^{\frac{n-1}{2}} (-1)^{M+1} \sum_{1 \le i_1 < \cdots < i_{2M} \le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right) \cdots \tan\left(\phi\left(\frac{x_{i_{2M}}}{2}\right)\right) \\
&\quad + \left[\sum_{M=0}^{\frac{n-3}{2}} (-1)^{M} \sum_{1 \le i_1 < \cdots < i_{2M+1} \le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right) \cdots \tan\left(\phi\left(\frac{x_{i_{2M+1}}}{2}\right)\right)\right] \tan\left(\phi\left(\frac{x_n}{2}\right)\right),
\end{aligned}$$

which yields $\dfrac{\sum_{M=0}^{\frac{n-3}{2}} (-1)^M \sum_{1 \le i_1 < \cdots < i_{2M+1} \le n} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right) \cdots \tan\left(\phi\left(\frac{x_{i_{2M+1}}}{2}\right)\right)}{1 + \sum_{M=1}^{\frac{n-1}{2}} (-1)^M \sum_{1 \le i_1 < \cdots < i_{2M} \le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right) \cdots \tan\left(\phi\left(\frac{x_{i_{2M}}}{2}\right)\right)}$

$= \dfrac{1}{\tan\left(\phi\left(\frac{x_n}{2}\right)\right)} = \cot\left(\phi\left(\frac{x_n}{2}\right)\right)$. By Lemma 2.2, we get $\tan\left(\phi\left(\frac{x_1}{2}\right) + \cdots + \phi\left(\frac{x_{n-1}}{2}\right)\right) = \tan\left(\frac{s\pi}{2} - \phi\left(\frac{x_n}{2}\right)\right)\ (s = 1, 3, \ldots, n-2)$. Thus, $\phi\left(\frac{x_1}{2}\right) + \cdots + \phi\left(\frac{x_n}{2}\right) = \frac{s\pi}{2}$ subject to $x_1 + \cdots + x_n = (n-2)\pi$, i.e., $\frac{x_1}{2} + \cdots + \frac{x_n}{2} = \frac{(n-2)\pi}{2}$.

By Lemma 2.1, we have $\phi\left(\frac{x}{2}\right) = k\left(\frac{x}{2} - \frac{(n-2)\pi}{2n}\right) + \frac{s\pi}{2n}$ for some fixed k belonging to the range $\max\left\{-\frac{s}{2}, \frac{s-n}{n-2}\right\} < k < \min\left\{\frac{s}{n-2}, \frac{n-s}{2}\right\}$. Therefore, $f(x) = \tan\left(k\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{s\pi}{2n}\right)\ \big(x \in I\big)$. □

4. Proof of Theorem 1.2

Let $f: I \to \mathbb{R} \setminus \{0\}$ satisfy (1.8) subject to the conditions (1.9) and (1.10). For a suitable bijection (to be determined) $\phi: I \to I$, put

(4.1) $f(x) = \tan(\phi(x))$ $(x \in I_1),\ \phi: I \to I$. From (1.8), we have

$$\begin{aligned}
0 &= \sum_{M=0}^{\frac{n-2}{2}} (-1)^M \sum_{1\le i_1<\cdots<i_{2M+1}\le n} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right)\cdots\tan\left(\phi\left(\frac{x_{i_{2M+1}}}{2}\right)\right)\\
&= \sum_{M=0}^{\frac{n-2}{2}} (-1)^M \sum_{1\le i_1<\cdots<i_{2M+1}\le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right)\cdots\tan\left(\phi\left(\frac{x_{i_{2M+1}}}{2}\right)\right)\\
&\quad+\left[\sum_{M=0}^{\frac{n-2}{2}} (-1)^M \sum_{1\le i_1<\cdots<i_{2M}\le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right)\cdots\tan\left(\phi\left(\frac{x_{i_{2M}}}{2}\right)\right)\right]\tan\left(\phi\left(\frac{x_n}{2}\right)\right)\\
&= \sum_{M=0}^{\frac{n-2}{2}} (-1)^M \sum_{1\le i_1<\cdots<i_{2M+1}\le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right)\cdots\tan\left(\phi\left(\frac{x_{i_{2M+1}}}{2}\right)\right)\\
&\quad+\left[1+\sum_{M=1}^{\frac{n-2}{2}} (-1)^M \sum_{1\le i_1<\cdots<i_{2M}\le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right)\cdots\tan\left(\phi\left(\frac{x_{i_{2M}}}{2}\right)\right)\right]\tan\left(\phi\left(\frac{x_n}{2}\right)\right),
\end{aligned}$$

which yields $\dfrac{\sum_{M=0}^{\frac{n-2}{2}} (-1)^M \sum_{1\le i_1<\cdots<i_{2M+1}\le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right)\cdots\tan\left(\phi\left(\frac{x_{i_{2M+1}}}{2}\right)\right)}{1+\sum_{M=1}^{\frac{n-2}{2}} (-1)^M \sum_{1\le i_1<\cdots<i_{2M}\le n-1} \tan\left(\phi\left(\frac{x_{i_1}}{2}\right)\right)\cdots\tan\left(\phi\left(\frac{x_{i_{2M}}}{2}\right)\right)}$ $= -\tan\left(\phi\left(\frac{x_n}{2}\right)\right)$. By Lemma 2.2, we get $\tan\left(\phi\left(\frac{x_1}{2}\right)+\cdots+\phi\left(\frac{x_{n-1}}{2}\right)\right) = \tan\left(\ell\pi - \phi\left(\frac{x_n}{2}\right)\right)$ $(\ell = 1, 2, \ldots, n-1)$. Thus, $\phi\left(\frac{x_1}{2}\right)+\cdots+\phi\left(\frac{x_n}{2}\right) = \ell\pi$ subject to $x_1 + \cdots + x_n = (n-2)\pi$, i.e., $\frac{x_1}{2} + \cdots + \frac{x_n}{2} = \frac{(n-2)\pi}{2}$. By Lemma 2.1, we have $\phi\left(\frac{x}{2}\right) = k\left(\frac{x}{2} - \frac{(n-2)\pi}{2n}\right) + \frac{\ell\pi}{n}$ for some fixed k belonging to the range $\max\left\{-\ell, \frac{2\ell-n}{n-2}\right\} < k < \min\left\{\frac{2\ell}{n-2}, \frac{n-2\ell}{2}\right\}$. Therefore, $f(x) = \tan\left(k\left(x - \frac{(n-2)\pi}{2n}\right) + \frac{\ell\pi}{n}\right)$ $(x \in I)$. □

Example 4.1 To determine the function $f: (0, \pi) \to \mathbb{R} \setminus \{0\}$ satisfying

(4.2) $f\left(\frac{x}{2}\right)f\left(\frac{y}{2}\right) + f\left(\frac{x}{2}\right)f\left(\frac{z}{2}\right) + f\left(\frac{y}{2}\right)f\left(\frac{z}{2}\right) = 1,$ $(x, y, z \in (0, \pi))$, subject to two conditions $x + y + z = \pi,$ $1 - f\left(\frac{x}{2}\right)f\left(\frac{y}{2}\right) - f\left(\frac{x}{2}\right)f\left(\frac{z}{2}\right) - f\left(\frac{y}{2}\right)f\left(\frac{z}{2}\right) \neq 0,$ we apply Theorem 1.1 to get $f(x) = \tan\left(k\left(x - \frac{\pi}{6}\right) + \frac{\pi}{6}\right)$ for all $x \in (0, \pi)$ and for some fixed $k \in (-1/2, 1)$. Finally, we have to check the validity of the solution so obtained. If $f(x) = \tan\left(k\left(x - \frac{\pi}{6}\right) + \frac{\pi}{6}\right)$, then

$$f\left(\frac{x}{2}\right)f\left(\frac{y}{2}\right)+f\left(\frac{x}{2}\right)f\left(\frac{z}{2}\right)+f\left(\frac{y}{2}\right)f\left(\frac{z}{2}\right)$$

$$=\tan\left(k\left(\frac{x}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)\tan\left(k\left(\frac{y}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)$$

$$+\tan\left(k\left(\frac{x}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)\tan\left(k\left(\frac{z}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)$$

$$+\tan\left(k\left(\frac{y}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)\tan\left(k\left(\frac{z}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)$$

$$=\tan\left(k\left(\frac{x}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)\tan\left(k\left(\frac{y}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)$$

$$+\tan\left(k\left(\frac{z}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)\bullet$$

$$\left[\tan\left(k\left(\frac{x}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)+\tan\left(k\left(\frac{y}{2}-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)\right]$$

$$=1.$$

Hence, the solution of the functional equation (4.2) is $f(x)=\tan\left(k\left(x-\frac{\pi}{6}\right)+\frac{\pi}{6}\right)$.

5. Conclusion and Discussion

Two functional equations that can be used to characterize the tangent function over a convex polygon are solved. The results so obtained contained most well-known identities about the tangent function over a triangle. An interesting problem, which seems totally nontrivial, is to ask for functional equations that can be used to characterize other trigonometric functions over a convex polygon.

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